5D Relativity and Solar System Tests of Braneworld Models

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Outline

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Extra Dimensions and the Quest for Unification
The Unification of Electricity and Magnetism

- **Maxwell** (1864),

$$\operatorname{div}\vec{B} = 0\,, \qquad \operatorname{rot}\vec{E} + \frac{1}{c}\,\partial_t\vec{B} = 0\,,$$
$$\operatorname{div}\vec{E} = 4\pi\varrho\,, \qquad \operatorname{rot}\vec{B} - \frac{1}{c}\,\partial_t\vec{E} = \frac{4\pi}{c}\,\vec{\jmath}\,.$$

- **Einstein** (1905), Special Theory of Relativity.
- **Minkowski** (1909), Geometrical Interpretation of Einstein's theory: Time is a 4^{th} dimension.
- Unified field equations of electromagnetism:

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0\,,$$
$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}\,j^\beta\,,$$

with the electromagnetic field tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

Extra Dimensions and the Quest for Unification

The Unification of Electromagnetism and Gravity

- Einstein-Maxwell Equations

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \kappa T^{\text{mat.}}_{\alpha\beta} + T^{\text{em.}}_{\alpha\beta} ,$$
$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 ,$$
$$\nabla_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta .$$

- **Kaluza** (1919), Introduction of a 5th dimension.

$$g_{AB} = \begin{pmatrix} g_{\alpha\beta} + \phi A_\alpha A_\beta & \phi A_\alpha \\ \phi A_\beta & \phi \end{pmatrix} , \quad \partial_4 g_{AB} = 0 .$$

- Ansatz for the unified field equation: $R_{AB} = 0$.
- With $\phi =$ const. and $F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha$ this yields

$$G_{\alpha\beta} = T^{\text{em.}}_{\alpha\beta} , \qquad \nabla_\alpha F^{\alpha\beta} = 0 , \qquad F_{\alpha\beta} F^{\alpha\beta} = 0 .$$

- Problem: $F_{\alpha\beta} F^{\alpha\beta} = 0 \rightleftharpoons \vec{E} = \vec{B}$.

Extra Dimensions and the Quest for Unification

The Unification of Electromagnetism and Gravity

- **Klein** (1926), Compactification of the 5th dimension.
- In 4D, the Einstein-Maxwell equations follow from a variational principal $\delta S = 0$ with

$$S = \int d^4x\sqrt{-g}\left[R + \frac{1}{8\pi}F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{c}j_\alpha A^\alpha\right] + S_{\text{mat.}}\,,$$

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha\,.$$

- Ansatz for the action in 5D:

$$S = \int d^5x\sqrt{-{}^{(5)}g}\;{}^{(5)}R = 2\pi\ell\int d^4x\sqrt{-{}^{(5)}g}\;{}^{(5)}R$$

$$= 2\pi\ell\int d^4x\sqrt{-g}\,\phi\left[R - \frac{\Box\phi}{\phi} + \frac{\phi}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\partial_\alpha\phi\,\partial^\alpha\phi}{2\phi^2}\right].$$

With $\phi = \frac{1}{2\pi} =$ const. this reduces to

$$S = \ell\int d^4x\sqrt{-g}\left[R + \frac{1}{8\pi}F_{\alpha\beta}F^{\alpha\beta}\right].$$

- Discovery of weak and strong interactions. More degrees of freedom. More dimensions.
- **Witten** (1981): A unified Kaluza-Klein theory of all fundamental interactions needs at least 11 dimensions.
- Several higher-dimensional programmes of unification:
  - Supergravity: 11D or less.
  - String theory: 26D (describes only bosons).
  - Five superstring theories: 10D (bosons and fermions).
- **Witten** (1994), M-Theory.
- Difference to the original idea of Kaluza and Klein: The gauge fields are *coupled to* the metric, rather than being *components of* the metric.

New Extra Dimensions

Large Extra Dimensions

- So far, all extra dimensions were compactified on Planck scale $\sim 10^{-35}$ m.
- **Wesson** and **Ponce de Leon** (1992), Space-Time-Matter Theory. 5D vacuum induces matter in 4D.
- **Arkani-Hamed**, **Dimopoulos** and **Dvali** (1998), two extra dimensions, compactified on $100\,\mu\text{m} \ldots 1\,\text{mm}$. Motivation: Hierarchy Problem. Modified Newtonian gravity: $1/r^2 \rightarrow 1/r^4$.
- **Randall** and **Sundrum** (1999), 5D Braneworld Szenario. Motivation: Hierarchy Problem. A warped extra dimension as an alternative to compactification.

Mathematics of Extra Dimensions
Dimensional Reduction

- Differential geometry works on manifolds with arbitrary dimensions.
- Suppose, our 4D spacetime Σ is a hypersurface, embedded in a 5D manifold $\mathcal{M}$,

$$\Phi: \quad \Sigma \to \mathcal{M}, \quad x^\alpha \mapsto X^A\left(x^0, x^1, x^2, x^3\right),$$

  with $\alpha = 0, 1, 2, 3$ and $A = 0, 1, 2, 3, 4$.
- The embedding Φ induces a metric on Σ,

$$g_{\alpha\beta} = g_{AB} e^A_\alpha e^B_\beta \quad \text{with} \quad e^A_\alpha \equiv \frac{\partial X^A}{\partial x^\alpha}.$$

- The Riemann Tensor can be defined on $\mathcal{M}$ *and* on Σ. They are related via the Gauß Equation,

$$R_{\alpha\beta\gamma\delta} = R_{ABCD} e^A_\alpha e^B_\beta e^C_\gamma e^D_\delta + K_{\alpha\gamma} K_{\delta\beta} - K_{\alpha\delta} K_{\gamma\beta}.$$

Mathematics of Extra Dimensions
Effective Field Equations

- Suppose, the 5D Einstein Equation on $\mathcal{M}$ takes the form

$$G_{AB} = -\Lambda_5 g_{AB} + \kappa_5 T_{AB} \,.$$

- Then, the Gauß Equation implies an effective 4D equation on Σ:

$$\begin{aligned} G_{\alpha\beta} = & -\tfrac{1}{2}\Lambda_5 g_{\alpha\beta} \\ & + \tfrac{2}{3}\kappa_5 \left[T_{AB} e^A_\alpha e^B_\beta + \left(T_{AB} n^A n^B - \tfrac{1}{4} {}^{(5)}T \right) g_{\alpha\beta} \right] \\ & + K K_{\alpha\beta} - K^\mu_\alpha K_{\mu\beta} + \tfrac{1}{2} g_{\alpha\beta} \left(K_{\mu\nu} K^{\mu\nu} - K^2 \right) \\ & - E_{\alpha\beta} \,. \end{aligned}$$

with $E_{\alpha\beta} \equiv C_{ABCD} e^A_\alpha n^B e^C_\beta n^D$.

Mathematics of Extra Dimensions
The Role of the Weyl Tensor

- The Weyl Tensor is the trace-free part of the Riemann Tensor,

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}\left(g_{a[c}R_{d]b} + R_{a[c}g_{d]b}\right) + \frac{2}{(n-1)(n-2)}\,{}^{(n)}R g_{a[c}g_{d]b}\,,$$

  with $a, b, c, d = 0, 1, \ldots, n-1$.
- The Ricci Tensor represents the part of the curvature that is determined by matter via

$$G_{ab} = \kappa_n T_{ab} \quad \rightleftharpoons \quad R_{ab} = \kappa_n \left[T_{ab} - \frac{1}{n-2}\,{}^{(n)}T g_{ab}\right].$$

- 4D vacuum: $R_{\alpha\beta} = 0 \curvearrowright R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}\,.$
- The Weyl Tensor represents the part of the gravitational field that propagates into vacuum.

Outline

Space-Time-Matter Theory
Basics

- Ansatz: 5D vacuum,

$$\Lambda_5 = 0\,, \quad T_{AB} = 0 \quad \curvearrowright \quad G_{AB} = 0\,.$$

- The effective 4D equation on Σ reduces to

$$G_{\alpha\beta} = KK_{\alpha\beta} - K^{\mu}_{\alpha}K_{\mu\beta} + \frac{1}{2}g_{\alpha\beta}\left(K_{\mu\nu}K^{\mu\nu} - K^2\right) - E_{\alpha\beta} \stackrel{!}{=} \kappa T_{\alpha\beta}\,.$$

- 4D matter from 5D geometry!
- Also called Induced-Matter Theory.

Space-Time-Matter Theory
Confinement Mechanism

- To confine matter on Σ, we need additional assumptions.
- e.g. the existence of a centripetal force F that obeys

$$n_A F^A = -K_{\alpha\beta} u^\alpha u^\beta \ .$$

- The source of this non-gravitational confining force is unclear.

Braneworld Szenarios
Confinement Mechanism

- Inspired by String/M-Theory.
- String/M-Theory deals with three fundamental objects:
  - Closed strings (e.g. the graviton)
  - Open strings (e.g. SM particles, gauge bosons)
  - Branes
- The ends of an open string are not free, but rather fixed on a brane!
- Braneworld idea:
  - Our 4D spacetime Σ is a brane, embedded in a higher dimensional manifold, called bulk.
  - All SM particles and gauge bosons are confined to Σ, except the graviton.
- Hierarchy Problem: Dilution of gravity.

Braneworld Szenarios
Field Equations

- 5D field equation: $G_{AB} = -\Lambda_5 g_{AB} + \kappa_5 T_{AB}$.
- Ansatz for the 5D energy-momentum tensor:

$$T_{AB} = \delta(X^4)\big[-\lambda g_{\alpha\beta} + T_{\alpha\beta}\big] e^{\alpha}_{A} e^{\beta}_{B} \,.$$

- Additional assumption: Z_2-Symmetry.
- Effective 4D field equation on the brane:

$$G_{\alpha\beta} = -\Lambda g_{\alpha\beta} + \kappa T_{\alpha\beta} + 6\tfrac{\kappa}{\lambda} S_{\alpha\beta} - E_{\alpha\beta}$$

  with

$$\kappa \equiv \tfrac{1}{6}\kappa_5^2 \lambda \,, \qquad \Lambda \equiv \tfrac{1}{2}\big(\Lambda_5 + \kappa\lambda\big) \,,$$

$$S_{\alpha\beta} \equiv \tfrac{1}{12} T T_{\alpha\beta} - \tfrac{1}{4} T_{\alpha\mu} T^{\mu}{}_{\beta} + \tfrac{1}{24} g_{\alpha\beta}\big(3 T_{\mu\nu} T^{\mu\nu} - T^2\big) \,.$$

- Two modifications to standard 4D Relativity:
  - A high-energy correction term $S_{\alpha\beta} \sim (T_{\alpha\beta})^2$.
  - A correction from the 5D Weyl tensor.

Outline

Brane Cosmology
Modified Friedmann Equations

- Cosmological principle (homogeneity, isotropy) on Σ. No extension to the bulk. Dependence on time and X^4.
- Ansatz: 5D metric in Gaussian normal coordinates

$$^{(5)}\mathrm{d}s^2 = -N^2(t,x)\mathrm{d}t^2 + A^2(t,y)\mathrm{d}\vec{x}^2 + \mathrm{d}y^2\,, \qquad y \equiv X^4\,.$$

- 5D field equations: $G_{AB} = -\Lambda_5 g_{AB} + \kappa_5 T_{AB}$ with

$$T_A^B(t,y) = \delta(y)\mathrm{diag}\big(-\varrho_\mathrm{b}\,, p_\mathrm{b}\,, p_\mathrm{b}\,, p_\mathrm{b}\,, 0\big).$$

- Solution: Modified Friedmann Equations on Σ

$$H_\mathrm{b}^2 \equiv \frac{\dot{a}_\mathrm{b}^2}{a_\mathrm{b}^2} = \frac{1}{6}\Lambda_5 + \frac{1}{36}\kappa_5^2\varrho_\mathrm{b}^2 + \frac{C}{a_\mathrm{b}^4}\,,$$

$$\dot{\varrho}_\mathrm{b} + 3H_\mathrm{b}(\varrho_\mathrm{b} + p_\mathrm{b}) = 0\,.$$

- 4D Standard Cosmology: $H^2 \sim \varrho\,, \frac{1}{a^2}\,.$

Brane Cosmology
Further Assumptions

- $H^2 \sim \varrho_{\rm b}^2$ ruins standard nucleosynthesis!
- Further assumptions:
  - Brane tension λ, $\varrho_{\rm b}(t) = \lambda + \varrho(t)$.
  - Negative bulk cosmological constant, $\Lambda = -6\kappa_5^2\lambda^2$.
- Modified Friedmann Equations on Σ

$$H_{\rm b}^2 = \frac{1}{18}\kappa_5^2\lambda\varrho + \frac{1}{36}\kappa_5^2\varrho^2 + \frac{C}{a_{\rm b}^4} .$$

- 4D Standard Cosmology is recovered approximatively with corrections:
  - ϱ^2, which dominates at high energies.
  - $C/a_{\rm b}^4$, called dark radiation.

Black Holes on the Brane
Effective Field Equations

- Assumptions:
  - 4D vacuum outside the black hole: $T_{\alpha\beta} = 0 \curvearrowright S_{\alpha\beta} = 0$.
  - 4D cosmological constant $\Lambda = 0 \curvearrowright \Lambda_5 = -\kappa\lambda < 0$.
  - 4D metric is static and spherical symmetric,

$$\mathrm{d}s^2 = -A\mathrm{d}t^2 + A^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,.$$

- Effective 4D field equations:

$$G_{\alpha\beta} = -E_{\alpha\beta}\,, \qquad E_{\alpha}{}^{\alpha} = 0 = R_{\alpha}{}^{\alpha}\,, \qquad \nabla^{\alpha}E_{\alpha\beta} = 0\,.$$

- Similar problem in standard 4D Relativity: $G_{\alpha\beta} = T^{\mathrm{em.}}_{\alpha\beta}$, with the free electromagnetic field ($j^{\alpha} = 0$).
- We can expect a Reissner-Nordström-type solution.

Black Holes on the Brane
Solution of the Effective Field Equations

- Decomposition of the projected Weyl tensor:

$$E_{\alpha\beta} = -\frac{2Q}{r^4}\left(u_\alpha u_\beta - r_\alpha r_\beta + \frac{1}{2}g_{\alpha\beta}\right).$$

- The solution is indeed of Reissner-Nordström type:

$$A = 1 - \frac{2GM}{r} + \frac{Q}{r^2}.$$

- Differences to the standard Reissner-Nordström solution:
  - No electric field on the brane. Q is the tidal charge parameter.
  - $A \sim Q$ instead of $\sim Q^2$. New physics for $Q < 0$!
  - $Q = Q(M)$, since M is the source of the 5D Weyl field.
- The gravitational field of M is reflected back to the brane by the negative 5D cosmological constant.

Black Holes on the Brane
Properties

- $Q = 0$: Schwarzschild case, one horizon at $r_S = 2GM$.
- $Q > 0$: Reissner-Nordström case with two horizons

$$r_\pm = GM\left(1 \pm \sqrt{1 - \frac{Q}{G^2M^2}}\right) , \qquad 0 < r_\pm < r_S \, .$$

Upper limit for Q: $Q_{\max} = G^2M^2$.

- New possibility, $Q < 0$: One horizon, lying outside the Schwarzschild horizon

$$r_+ = GM\left(1 + \sqrt{1 - \frac{Q}{G^2M^2}}\right) > r_S \, .$$

Black hole with lower temperature and greater entropy.

Outline

Solar System Tests

- Search for further constraints of Q from astronomical observations.
- Use the classical solar system tests of 4D Relativity:
  - Light deflection
  - Perihelion precession

Light Deflection

- 4D GR predicts the deflection of a light ray passing near a Schwarzschild star by an angle

$$\varphi_{\rm S} = \frac{4GM}{c^2 R}\,.$$

- For a Braneworld black hole the deflection angle is

$$\varphi_{\rm BW} = \varphi_{\rm S}\left(1 - \frac{Q}{R^2}\right).$$

- Observed value: $\varphi_{\rm obs.} = \varphi_{\rm S}\left(1 + \Delta\right)$ with $\Delta \leq 0.0017$.
- This implies that Q should be of the order

$$|Q| \leq 8 \times 10^{18}\,{\rm cm}^2.$$

Perihelion Precession

- Perihelion = smallest distance of a planet to the sun.
- 4D GR predicts perihelion precession $\delta\varphi \equiv \Delta\varphi - 2\pi$.
- For planet Mercury, orbiting a Schwarzschild star:

$$\delta\varphi_{\mathrm{S}} = \frac{6\pi GM}{c^2 a(1-e^2)} = 42.94'' \text{ per century.}$$

- Observed value: $\delta\varphi_{\mathrm{obs.}} = (43.11 \pm 0.21)''$ per century.
- Difference: $\delta\varphi_{\mathrm{obs.}} - \delta\varphi_{\mathrm{S}} = 0.17''$ per century.
- Perihelion shift around a Braneworld black hole:

$$\delta\varphi_{\mathrm{BW}} = \delta\varphi_{\mathrm{S}} - \frac{\pi c^2 Q}{GMa(1-e^2)}.$$

- This implies that Q should be of the order

$$|Q| \leq \frac{GMa(1-e^2)}{\pi c^2} \cdot 0.17'' = 5.17 \times 10^8 \,\mathrm{cm}^2.$$

Outline

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