Compact Stars in the Braneworld

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Outline

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Tolman-Oppenheimer-Volkoff equations

- Our goal is to solve the Einstein equations

$$G_{\alpha\beta} := R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

for a static and spherical star.

- Ansatz for the line element

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\,\mathrm{d}t^2 + \mathrm{e}^{2\Lambda(r)}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right).$$

- Stellar matter modelled as a perfect fluid

$$T_\alpha^\beta = \begin{pmatrix} -\varrho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

Tolman-Oppenheimer-Volkoff equations

- This yields the TOV equations of stellar structure

$$
\begin{aligned}
m' &= 4\pi r^2 \varrho\,, \\
p' &= -(\varrho + p)\Phi'\,, \\
\Phi' &= \frac{Gm + 4\pi G r^3 p}{r(r-2Gm)}
\end{aligned}
$$

with the line element

$$
\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\,\mathrm{d}t^2 + \left(1 - \frac{2Gm(r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right).
$$

- 3 equations, 4 unknown functions $m(r)$, $\varrho(r)$, $p(r)$ and $\Phi(r)$.
- In order to close the system, we need an equation of state

$$
p = p(\varrho).
$$

Solutions of the TOV equations

- **Exterior**: $\varrho = 0$, $p = 0$ and

$$m' = 0\,, \qquad \Phi' = \frac{Gm}{r(r-2Gm)}\,, \quad \curvearrowright \quad m = \text{const} =: M\,, \qquad e^{2\Phi} = 1 - \frac{2GM}{r}\,.$$

- This leads to the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right).$$

Solutions of the TOV equations

- **Interior**: An analytical solution exists only for the ideal case of a homogeneous density star with

$$\varrho = \text{const} =: \varrho_* \quad \text{for all} \quad p(r)\,.$$

- The TOV equations imply the internal Schwarzschild solution

$$\mathrm{d}s^2 = -\left(\frac{3}{2}\sqrt{1-\frac{2GM}{R}} - \frac{1}{2}\sqrt{1-\frac{2GMr^2}{R^3}}\right)^2 \mathrm{d}t^2 + \left(1-\frac{2GMr^2}{R^3}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right).$$

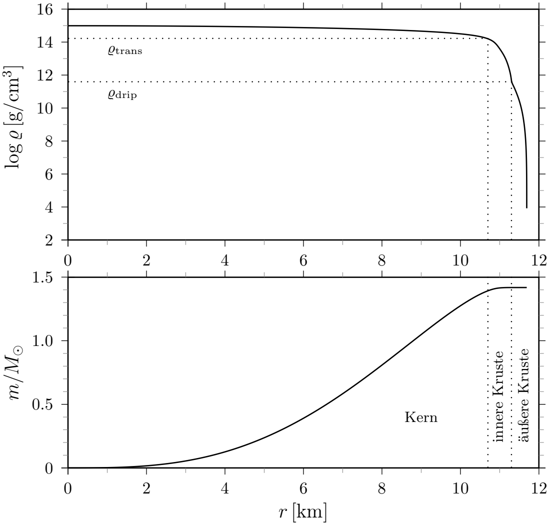
- Interesting property of this solution:

$$\frac{GM}{R} < \frac{4}{9}\,, \quad R_{\max} = \left(3\pi G\varrho_*\right)^{-1/2}, \quad M_{\max} = \tfrac{4}{9}\left(3\pi G^3\varrho_*\right)^{-1/2}.$$

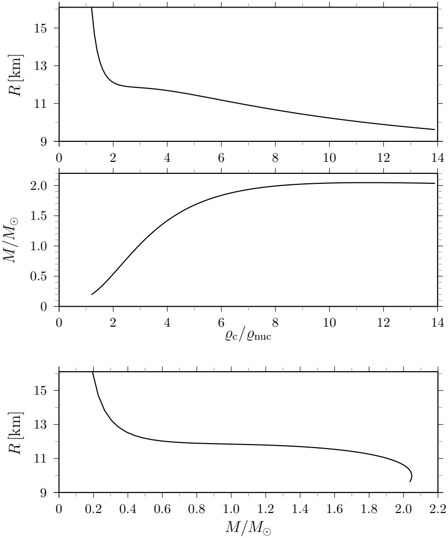
Solutions of the TOV equations

- For realistic stars with inhomogenious density, there are no analytical solutions.
- We have to use numerical methods.
- The TOV equations are a system of first order, ordinary differential equations.
- They can be solved numerically using a Runge-Kutta-Algorithm.
- For the numerical calculations, I used the equation of state SLy for the neutron star matter.

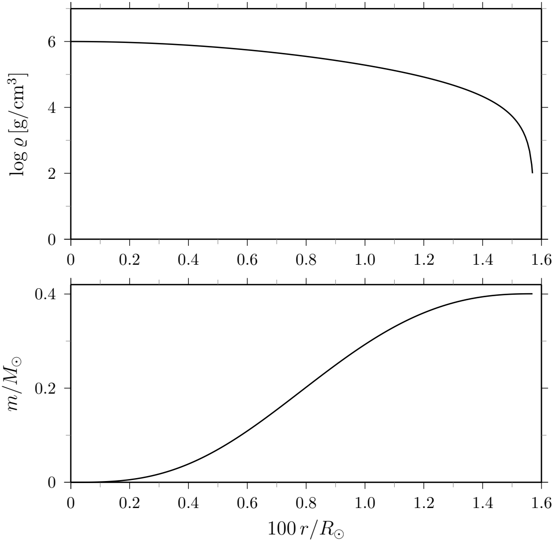
Neutron Stars



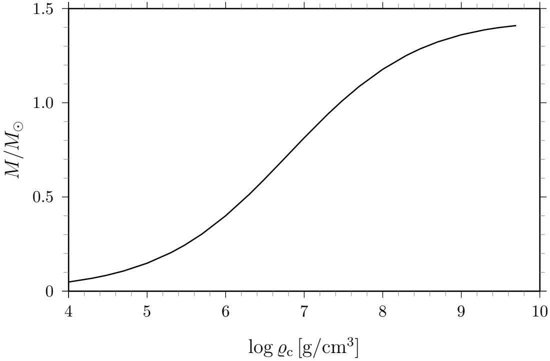
Neutron Stars



White Dwarfs



White Dwarfs



Outline

Introduction

- The braneworld paradigm is based on 3 assumptions:
  1. 4D spacetime is a hypersurface of a 5D manifold, called bulk.
  2. Gravity is a 5D phenomenom. The field equations in 5D have the same form as Einstein's equations in 4D:

$$G_{AB} = -\Lambda_5 g_{AB} + \kappa_5 T_{AB} \,.$$

  3. Matter and the fundamental interactions (except gravity) are confined to the hypersurface. Inspired by M-/string theory, the hypersurface is called a brane. The confinement of matter enters phenomenologically via

$$T_{AB} = \delta(\chi)\big[-\lambda \widehat{g}_{AB} + \widehat{T}_{AB}\big] \,.$$

- Due to the negative cosmological constant Λ_5, gravity is effectively localised near the brane.
- The 5^{th} dimension is not compactified, but can be infinitely large.

Effective Field Equations on the Brane

- On the brane an effective 4D gravity is induced by the 5D field equations:

$$G_{\alpha\beta} = -\Lambda g_{\alpha\beta} + \kappa T_{\alpha\beta} + \tfrac{6\kappa}{\lambda} S_{\alpha\beta} - E_{\alpha\beta}$$

  with

$$\begin{aligned}
\Lambda &:= \tfrac{1}{2}\left(\Lambda_5 + \tfrac{1}{6}\kappa_5^2\lambda^2\right), \\
\kappa &:= \tfrac{1}{6}\kappa_5^2\lambda, \\
S_{\alpha\beta} &:= \tfrac{1}{12} T T_{\alpha\beta} - \tfrac{1}{4} T_{\alpha\mu} T^{\mu}{}_{\beta} + \tfrac{1}{24} g_{\alpha\beta}\left(3T_{\mu\nu}T^{\mu\nu} - T^2\right).
\end{aligned}$$

- In the limit

$$\lambda \to \infty\,, \qquad \kappa_5 \to 0\,, \qquad E_{\alpha\beta} \to 0$$

  (with finite κ) gravity is localised on the brane and General Relativity is recovered.

Outline

Brane-TOV equations

- Now, we want to solve the modified Einstein equations

$$G_{\alpha\beta} = \kappa T^{\text{eff}}_{\alpha\beta}\,, \qquad T^{\text{eff}}_{\alpha\beta} := T_{\alpha\beta} + \tfrac{6}{\lambda} S_{\alpha\beta} - \tfrac{1}{\kappa} E_{\alpha\beta}$$

  for a static and spherical star on the brane.
- Ansatz for the line element

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\,\mathrm{d}t^2 + \mathrm{e}^{2\Lambda(r)}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right).$$

- The brane metric, and also the Einstein tensor $G_{\alpha\beta}$ are the same as in 4D.

Brane-TOV equations

- Stellar matter modelled as a perfect fluid

$$T_\alpha^\beta = \begin{pmatrix} -\varrho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} .$$

- This implies the high-energy correction

$$S_\alpha^\beta = \frac{1}{12} \begin{pmatrix} -\varrho^2 & 0 & 0 & 0 \\ 0 & (\varrho^2+2\varrho p) & 0 & 0 \\ 0 & 0 & (\varrho^2+2\varrho p) & 0 \\ 0 & 0 & 0 & (\varrho^2+2\varrho p) \end{pmatrix} .$$

- The static and spherically symmetric projected Weyl tensor can be written as

$$-\frac{1}{\kappa} E_\alpha^\beta = \begin{pmatrix} -\mathcal{U} & 0 & 0 & 0 \\ 0 & \frac{1}{3}(\mathcal{U}+2\mathcal{P}) & 0 & 0 \\ 0 & 0 & \frac{1}{3}(\mathcal{U}-\mathcal{P}) & 0 \\ 0 & 0 & 0 & \frac{1}{3}(\mathcal{U}-\mathcal{P}) \end{pmatrix} .$$

Brane-TOV equations

- We end up with the effective energy-momentum tensor

$$T_{\alpha}^{\beta\,\mathrm{eff}} = \begin{pmatrix} -\varrho^{\mathrm{eff}} & 0 & 0 & 0 \\ 0 & p_{\perp}^{\mathrm{eff}} & 0 & 0 \\ 0 & 0 & p_{\parallel}^{\mathrm{eff}} & 0 \\ 0 & 0 & 0 & p_{\parallel}^{\mathrm{eff}} \end{pmatrix},$$

with

$$\varrho^{\mathrm{eff}} := \varrho + \frac{\varrho^2}{2\lambda} + \mathcal{U}\,,$$

$$p_{\perp}^{\mathrm{eff}} := p + \frac{p\varrho}{\lambda} + \frac{\varrho^2}{2\lambda} + \frac{\mathcal{U} + 2\mathcal{P}}{3}\,,$$

$$p_{\parallel}^{\mathrm{eff}} := p + \frac{p\varrho}{\lambda} + \frac{\varrho^2}{2\lambda} + \frac{\mathcal{U} - \mathcal{P}}{3}\,.$$

Brane-TOV equations

- The brane-TOV equations follow from $G_{\alpha\beta} = \kappa T^{\text{eff}}_{\alpha\beta}$:

$$
\begin{aligned}
\widetilde{m}' &= 4\pi r^2 \left(\varrho + \frac{\varrho^2}{2\lambda} + \mathcal{U} \right) , \\
p' &= -(\varrho + p)\Phi' , \\
\Phi' &= \frac{G\widetilde{m} + 4\pi G r^3 \left[p + \frac{p\varrho}{\lambda} + \frac{\varrho^2}{2\lambda} + \frac{1}{3}(\mathcal{U} + 2\mathcal{P}) \right]}{r(r - 2G\widetilde{m})} , \\
\mathcal{U}' &= -2\mathcal{P}' - \frac{6\mathcal{P}}{r} - (4\mathcal{U} + 2\mathcal{P})\Phi' - 3\frac{\varrho + p}{\lambda}\varrho'
\end{aligned}
$$

with the line element

$$
\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\,\mathrm{d}t^2 + \left(1 - \frac{2G\widetilde{m}(r)}{r} \right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2 \right).
$$

Brane-TOV equations

- 4 equations, 6 unknown functions $\widetilde{m}(r)$, $\varrho(r)$, $p(r)$, $\Phi(r)$ and the Weyl terms $\mathcal{U}$ and $\mathcal{P}$.
- In order to close the system, we need an equation of state $p = p(\varrho)$ plus an additional relation between $\mathcal{U}$ and/or $\mathcal{P}$.
- I assume, that the Weyl terms obey an equation-of-state like relation $\mathcal{P} = w\mathcal{U}$.
- Then, the brane-TOV equations read

$$\widetilde{m}' = 4\pi r^2\left(\varrho + \frac{\varrho^2}{2\lambda} + \mathcal{U}\right),$$

$$p' = -(\varrho + p)\Phi',$$

$$\Phi' = \frac{G\widetilde{m} + 4\pi G r^3\left[p + \frac{p\varrho}{\lambda} + \frac{\varrho^2}{2\lambda} + \frac{1+2w}{3}\mathcal{U}\right]}{r(r - 2G\widetilde{m})},$$

$$\mathcal{U}' = -\frac{3}{1+2w}\left[2w\mathcal{U}r^{-1} + \tfrac{2}{3}(2+w)\mathcal{U}\Phi' + \tfrac{1}{\lambda}(\varrho + p)\varrho'\right].$$

Analytical Exterior Solution for $w = -2$

- In the case $w = -2$ the exterior brane-TOV equations can be solved analytically.
- **Exterior**: $\varrho = 0$, $p = 0$ and

$$\begin{aligned} \widetilde{m}' &= 4\pi r^2 \mathcal{U}\,, \\ \Phi' &= \frac{G\widetilde{m} - 4\pi G r^3 \mathcal{U}}{r(r-2G\widetilde{m})}\,, \\ \mathcal{U}' &= -\frac{4\mathcal{U}}{r}\,, \end{aligned} \quad \curvearrowright \quad \begin{aligned} \widetilde{m}(r) &= \widetilde{m}(R) + 4\pi\mathcal{U}(R)R^4\left(\frac{1}{R} - \frac{1}{r}\right) \\ e^{2\Phi} &= 1 - \frac{2G\widetilde{m}(r)}{r}\,, \\ \mathcal{U}(r) &= \mathcal{U}(R)\left(\frac{R}{r}\right)^4. \end{aligned}$$

- If $\mathcal{U}(R) \neq 0$, then $\widetilde{m}(r) \neq$ const in the exterior!

Analytical Exterior Solution for $w = -2$

- This leads to a Reissner-Nordström like solution

$$\mathrm{d}s^2 = -\left(1 - \frac{2G\widetilde{m}_\infty}{r} + \frac{Q}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2G\widetilde{m}_\infty}{r} + \frac{Q}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)$$

with

$$\widetilde{m}_\infty := \widetilde{m}(R) + 4\pi\mathcal{U}(R)R^3\,, \qquad Q := \kappa\,\mathcal{U}(R)R^4\,.$$

- A numerical calculation shows, that for $w = -2$ we have in general $\mathcal{U}(R) < 0$.
- That means, the effective mass tends to $\widetilde{m}_\infty < \widetilde{m}(R)$.

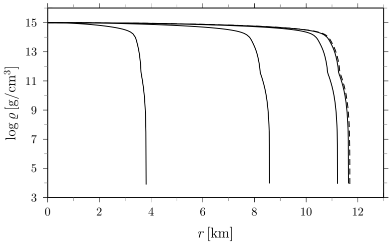
Neutron Stars on the Brane

- As in the relativistic case, the brane-TOV equations can be solved numerically.
- I used the same equation of state (SLy) as in the relativistic case, taking into account the inhomogeneous energy density inside the neutron star.
- The following pages show the results of the numerical integration for neutron stars.

Neutron Stars on the Brane

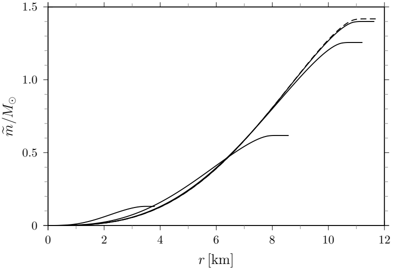
$w = -100$

$\lambda\,[\text{dyn/cm}^2] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



$w = -100$

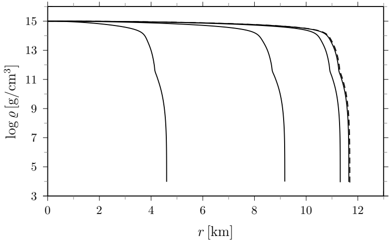
$\lambda\,[\text{dyn/cm}^2] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



Neutron Stars on the Brane

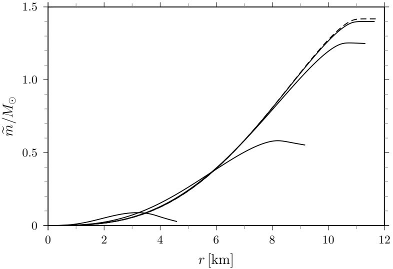
$w = -2$

$\lambda\,[\mathrm{dyn/cm^2}] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



$w = -2$

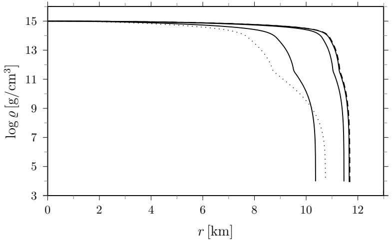
$\lambda\,[\mathrm{dyn/cm^2}] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



Neutron Stars on the Brane

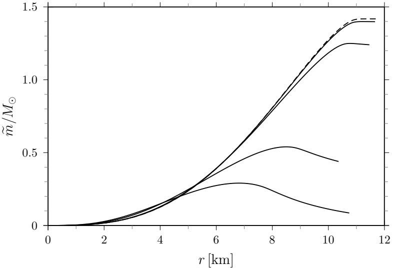
$w = -1$

$\lambda\,[\text{dyn/cm}^2] = 5 \cdot 10^{35}$ (dotted), $10^{36},\ 10^{37},\ 10^{38}$



$w = -1$

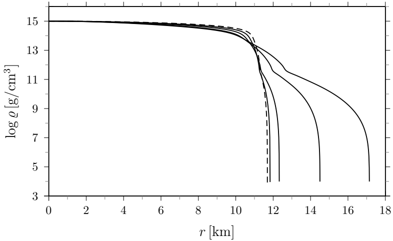
$\lambda\,[\text{dyn/cm}^2] = 5 \cdot 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



Neutron Stars on the Brane

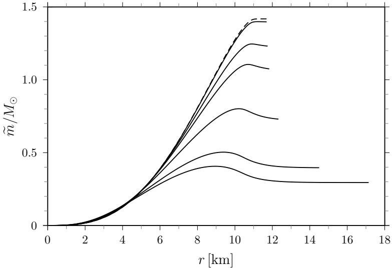
$w = -0{,}6$

$\lambda\,[\text{dyn}/\text{cm}^2] = 5\cdot 10^{36},\ 2\cdot 10^{36},\ 10^{36},\ 8\cdot 10^{35}$



$w = -0{,}6$

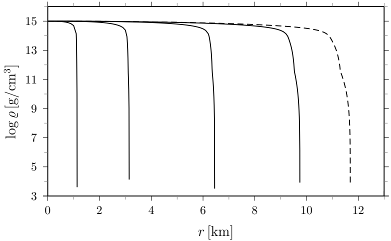
$\lambda\,[\text{dyn}/\text{cm}^2] = 8\cdot 10^{35},\ 10^{36},\ 2\cdot 10^{36},\ 5\cdot 10^{36},\ 10^{37},\ 10^{38}$



Neutron Stars on the Brane

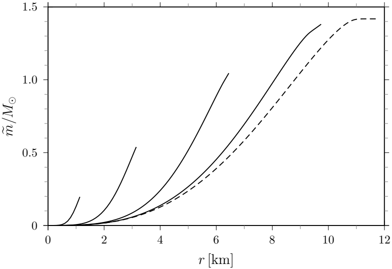
$w = -0{,}2$

$\lambda\,[\mathrm{dyn/cm^2}] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



$w = -0{,}2$

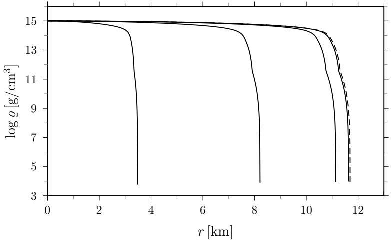
$\lambda\,[\mathrm{dyn/cm^2}] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



Neutron Stars on the Brane

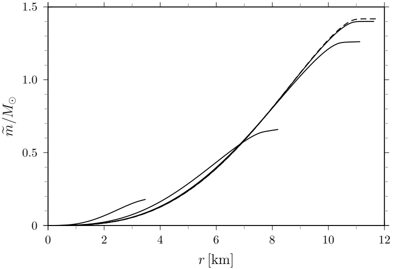
$w = 2$

$\lambda\,[\mathrm{dyn/cm^2}] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



$w = 2$

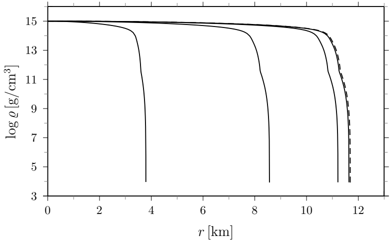
$\lambda\,[\mathrm{dyn/cm^2}] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



Neutron Stars on the Brane

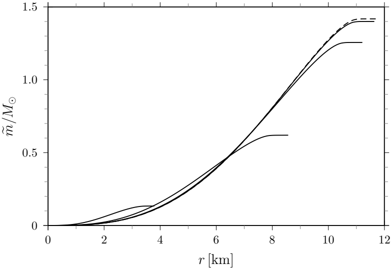
$w = 100$

$\lambda\,[\mathrm{dyn/cm^2}] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



$w = 100$

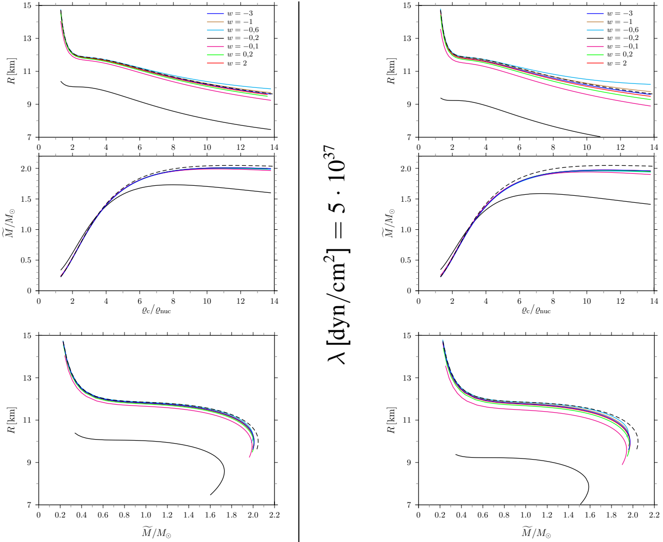
$\lambda\,[\mathrm{dyn/cm^2}] = 10^{35},\ 10^{36},\ 10^{37},\ 10^{38}$



Neutron Stars on the Brane

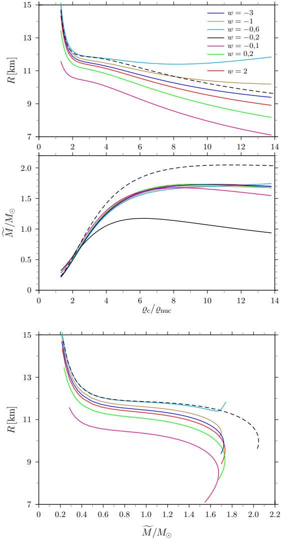
$\lambda\,[\mathrm{dyn/cm^2}] = 10^{38}$

$\lambda\,[\mathrm{dyn/cm^2}] = 5 \cdot 10^{37}$

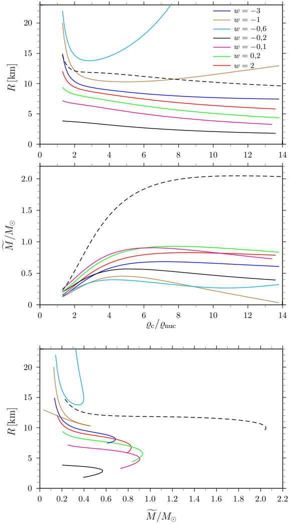


Neutron Stars on the Brane

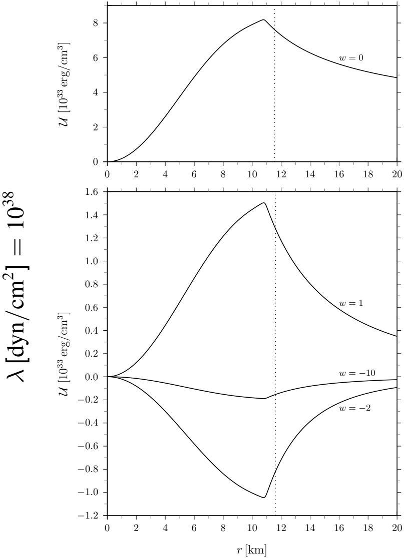
$\lambda\,[\mathrm{dyn/cm^2}] = 10^{37}$



$\lambda\,[\mathrm{dyn/cm^2}] = 10^{36}$

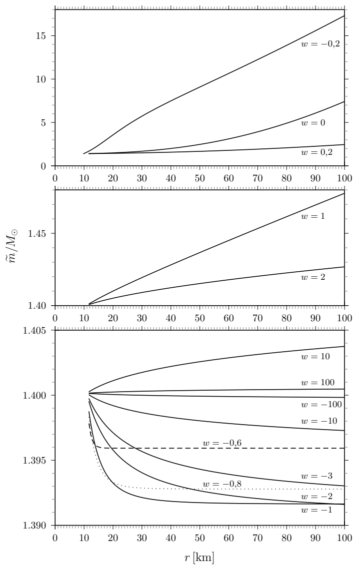


Neutron Stars on the Brane

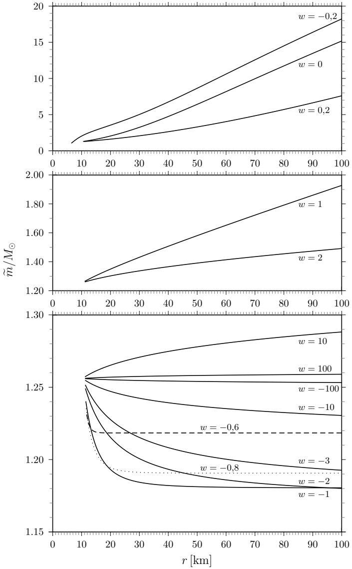


Neutron Stars on the Brane

λ [dyn/cm^2] $= 10^{38}$

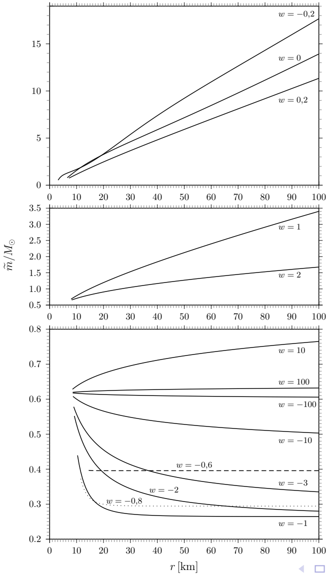


λ [dyn/cm^2] $= 10^{37}$



Neutron Stars on the Brane

$\lambda\,[\mathrm{dyn/cm^2}] = 10^{36}$



White Dwarfs on the Brane

- Basically, there is no difference between conventional and braneworld white dwarfs.
- Consider the effective density

$$\varrho^{\text{eff}} = \varrho + \Delta\varrho\,, \qquad \Delta\varrho := \frac{c^2\varrho^2}{2\lambda} + \frac{\mathcal{U}}{c^2}$$

  and the parameters $\lambda = 10^{38}\,\text{dyn}/\text{cm}^2$ and $w = -2$.
- For a white dwarf with $\varrho_\text{c} = 10^6\,\text{g}/\text{cm}^3$ the relative density correction $\Delta\varrho/\varrho$ is $4{,}5 \cdot 10^{-12}$ at the center and $4{,}1 \cdot 10^{-13}$ at $\mathcal{U}_\text{min}$.
- For a neutron star with $\varrho_\text{c} = 10^{15}\,\text{g}/\text{cm}^3$ the relative density correction $\Delta\varrho/\varrho$ is $4{,}5 \cdot 10^{-3}$ at the center and -1% at $\mathcal{U}_\text{min}$.

Outline

Conclusions

- For white dwarfs there are basically no differences to general relativity.
- For neutron stars the relative corrections of mass and density are bigger than 10^{-3} if $\lambda \lesssim 10^{39}\,\mathrm{dyn/cm^2}$.
- A brane tension $\lambda < 5 \cdot 10^{36}\,\mathrm{dyn/cm^2}$ is in contradiction with observed neutron star masses.
- Neutron stars on the brane are in general more compact: They have smaller radii (except for $-1 \lesssim w < -0{,}5$) and are less massive (except for $-0{,}5 < w \lesssim -0{,}1$ and $\varrho_c/\varrho_{\mathrm{nuc}} \lesssim 4$).
- A generic property of the brane-TOV equations are exterior solutions with $\widetilde{m} \neq \mathrm{const}$.

Outline

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