Supersymmetry in a classical world: new insights on stochastic dynamics from topological field theory

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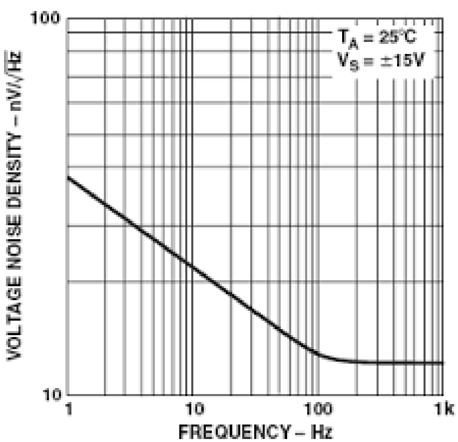
Plan of the Talk

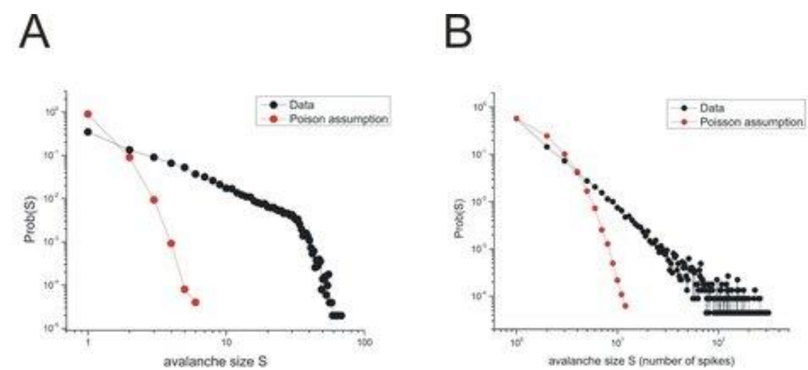
- Ubiquitous Chaotic Long-Range Order and State-of-Art Theory of Stochastic Differential Equations (SDEs) and Dynamical Systems (DS) Theory
- The Cohomological Theory of SDEs (ChT-SDE): Extended Hilbert Space; Topological Supersymmetry and Its Spontaneous Breaking (Chaos); Ergodicity and Time Reversal Symmetry Breaking; Phase Diagram and Noise-Induced Chaos.
- Example: Healthy Brain is at the Phase of Noise-Induced Chaos
- Conclusion

Mysterious Chaotic Long-Range Order

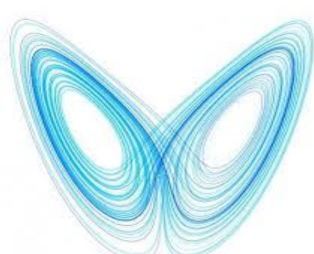
1/f, pink, or flicker noise (long-term memory effect)- algebraic power-spectra

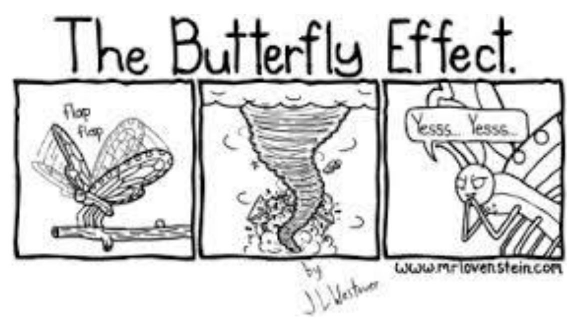


Power-law (or scale free) statistics (e.g, Richter Scale) of highly nonlinear processes or events such as earthquakes



Butterfly Effect, i.e., infinite memory of perturbations/init.cond.





Mysterious Chaotic Long-Range Order

Darwin Theory of Evolution (1859)

Punctuated Equilibria (Eldredge, Gauld, 1972)

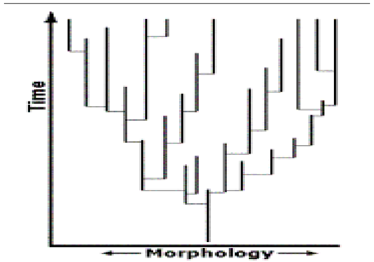
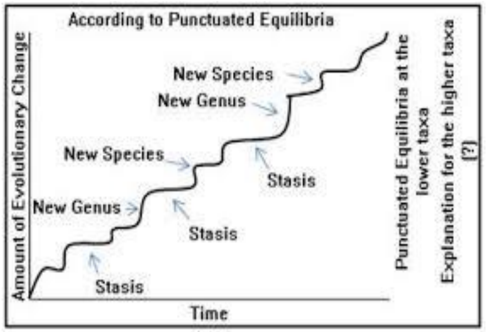
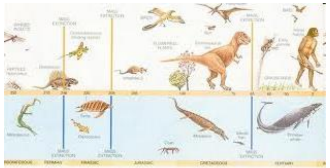


Figure 2. An evolutionary tree under a "punctuated equilibrium" model.[18] Most change in morphology (i.e. macroevolutionary change) takes place over geologically short time periods in small populations, such that transitional forms are unlikely to be fossilized.

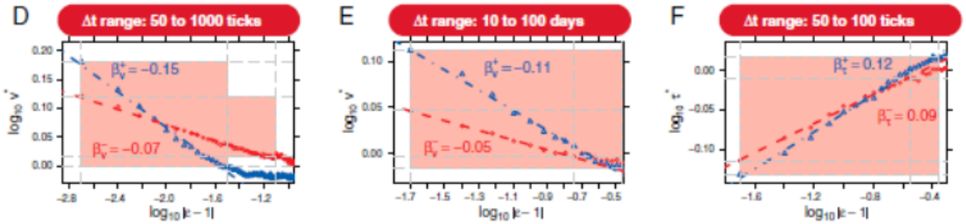


Biological evolution



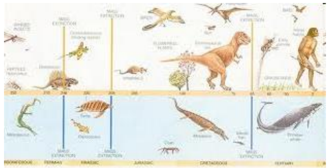
Switching processes in financial markets

Tobias Preis[a,b,c,1], Johannes J. Schneider[d], and H. Eugene Stanley[a,1]



Econodynamics

Biological evolution

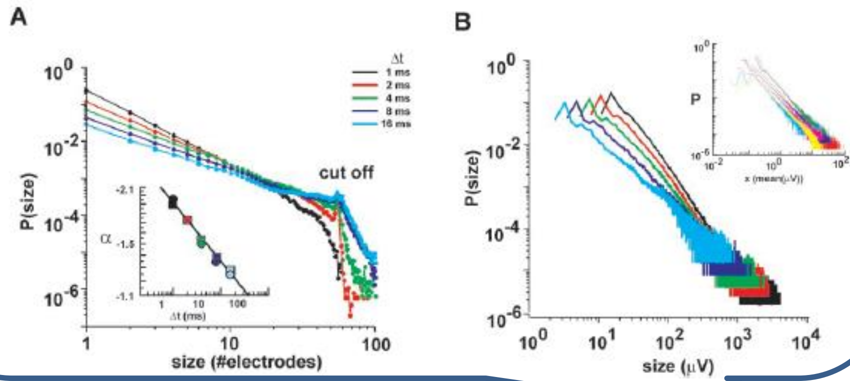


Mysterious Chaotic Long-Range Order

Neuronal Avalanches in Neocortical Circuits

John M. Beggs and Dietmar Plenz

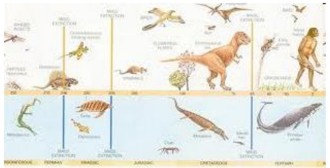
Unit of Neural Network Physiology, Laboratory of Systems Neuroscience, National Institute of Mental Health, Bethesda, Maryland 20892



Neurophysics

Econodynamics

Biological evolution



Mysterious Chaotic Long-Range Order

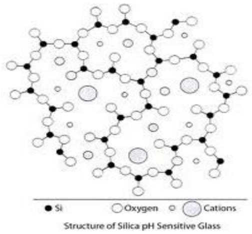
Traffic

Flocking

Internet

Geophysics

Soft condensed matter

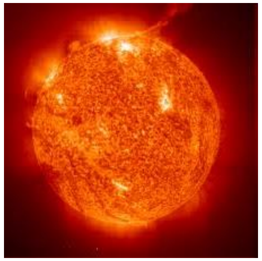


Aerodynamics

Neurophysics

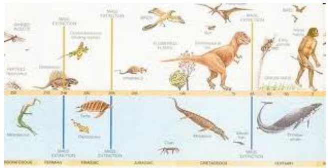
Econodynamics

Magneto-hydrodynamics



Hydrodynamics

Biological evolution



Mysterious Chaotic Long-Range Order

Astrophysics

Traffic

Flocking

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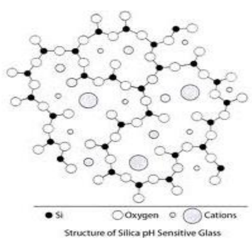

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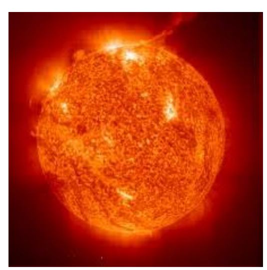
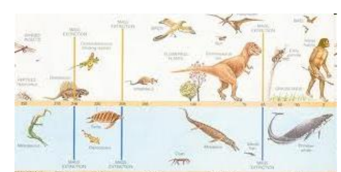


Hydrodynamics

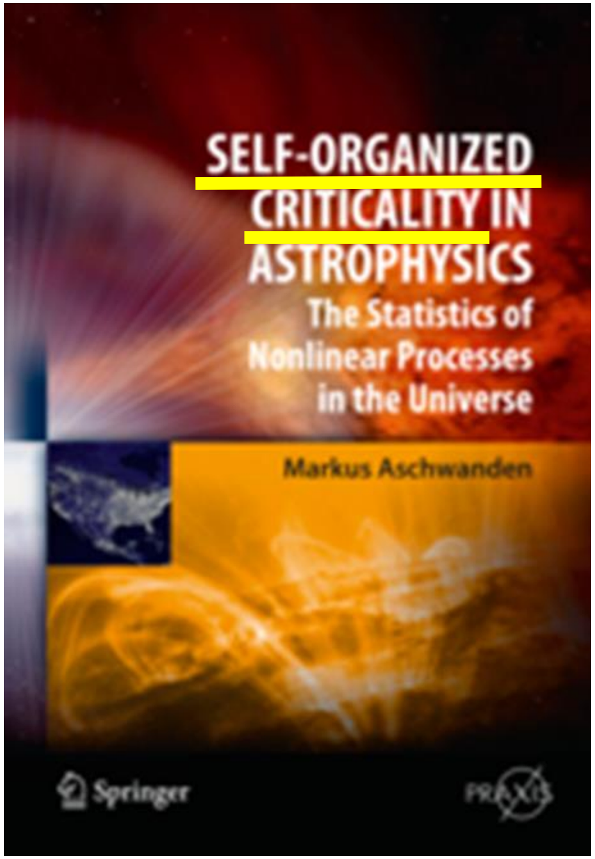
Biological evolution



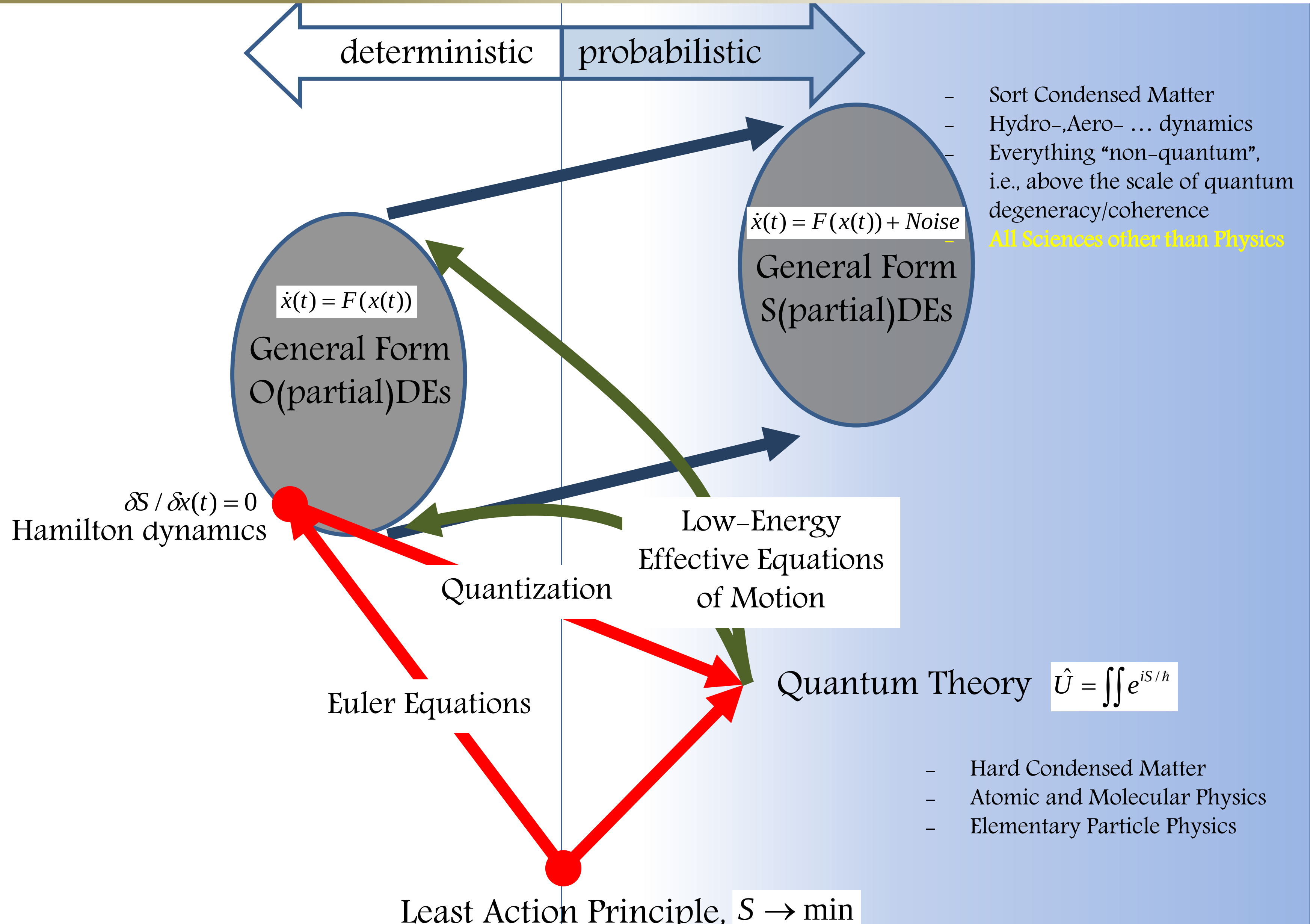
Mysterious Chaotic Long-Range Order: Astrophysics

Astrophysics:

… a wide range of phenomena in astrophysics, such as planetary magnetospheres, solar flares, cataclysmic variable stars, accretion disks, black holes and gamma-ray bursts, and also to phenomena in galactic physics and cosmology…



Generality of S(partial)DEs in Physics



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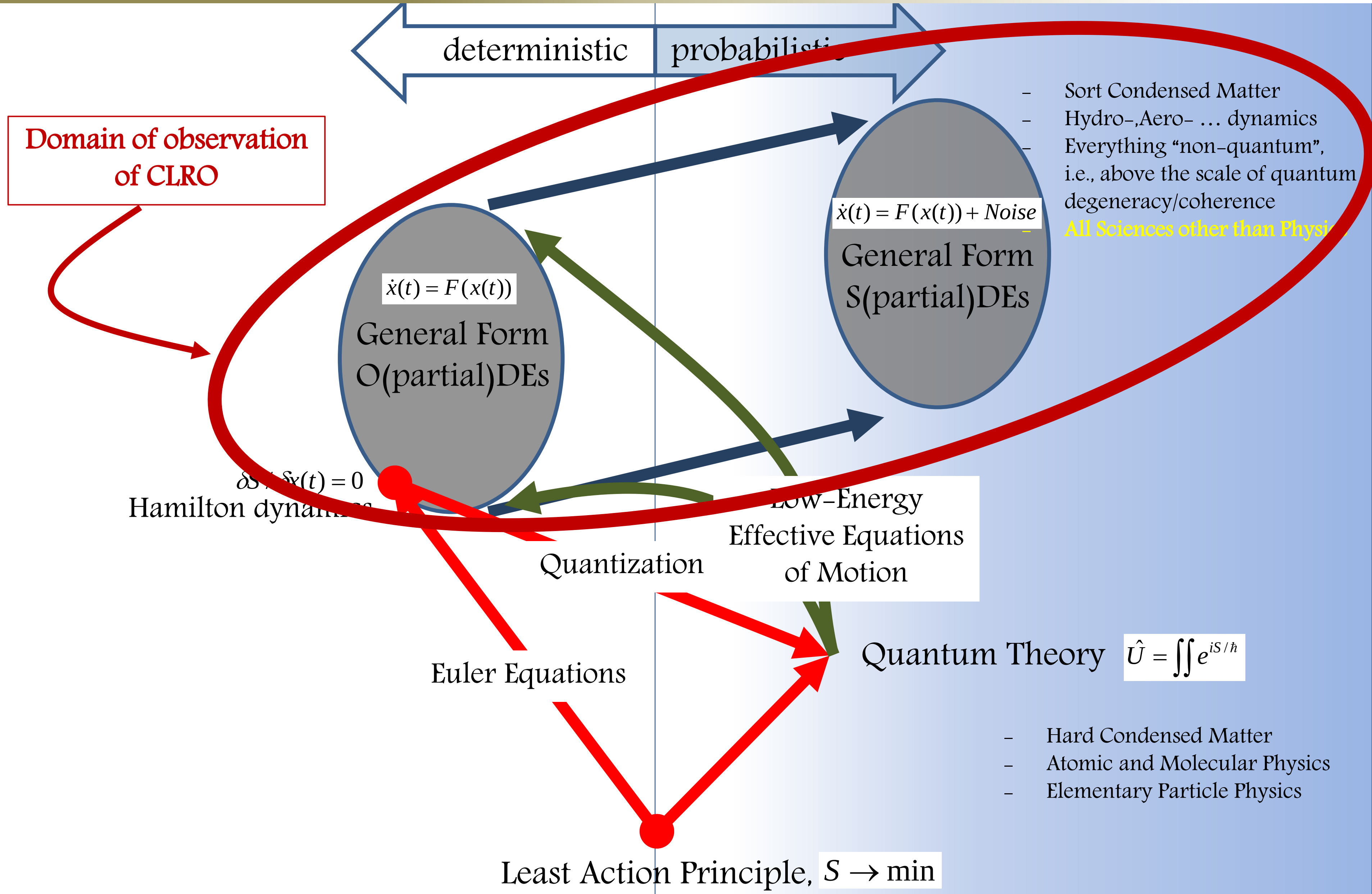
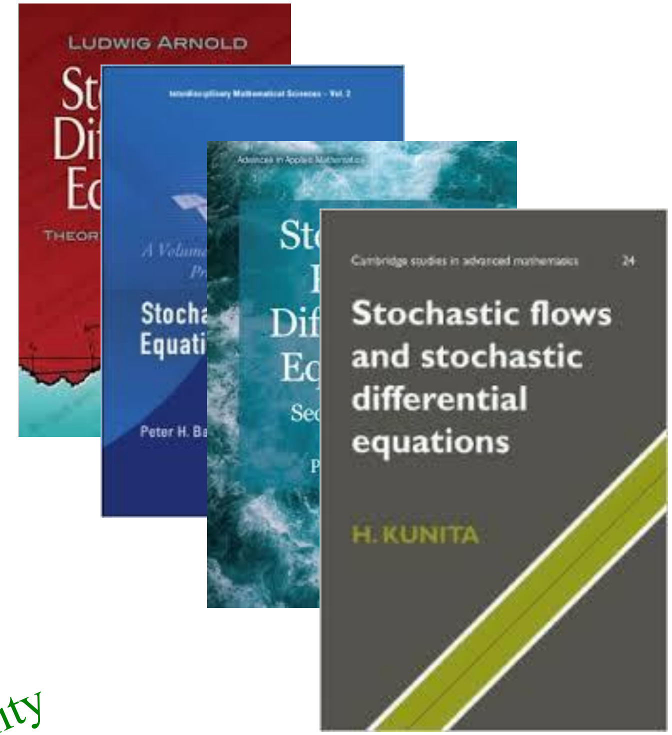


State-of-Art theory of S(partial)DEs

Theory of SDEs is older than the quantum theory and general relativity.

Theories of Brownian motion: Smoluchowskii (1906), Einstein (1905), even earlier works



1/f noise

Intermittency

Self-Organization

Dynamical Chaos

Ergodicity

Complexity

Turbulence

Self-Organized Criticality

Non-Equilibrium Dynamics

Modern Picture: Open Questions

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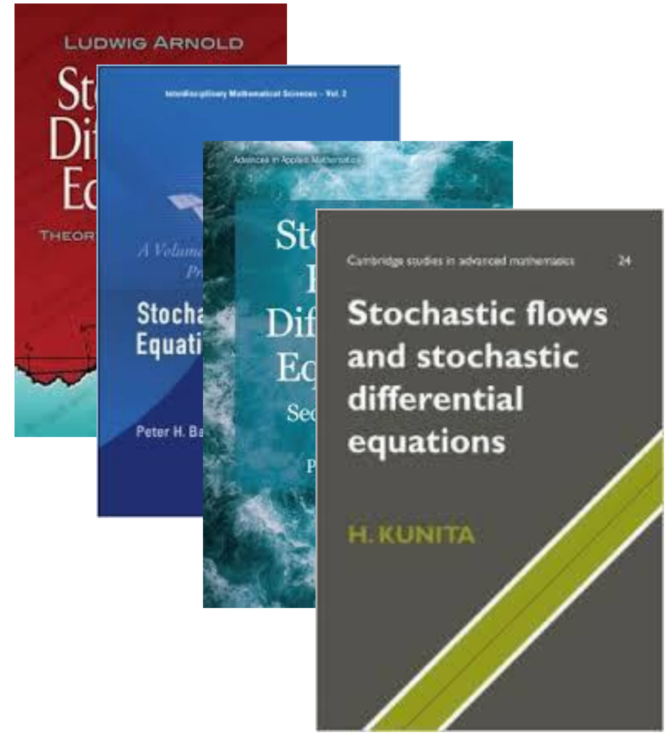
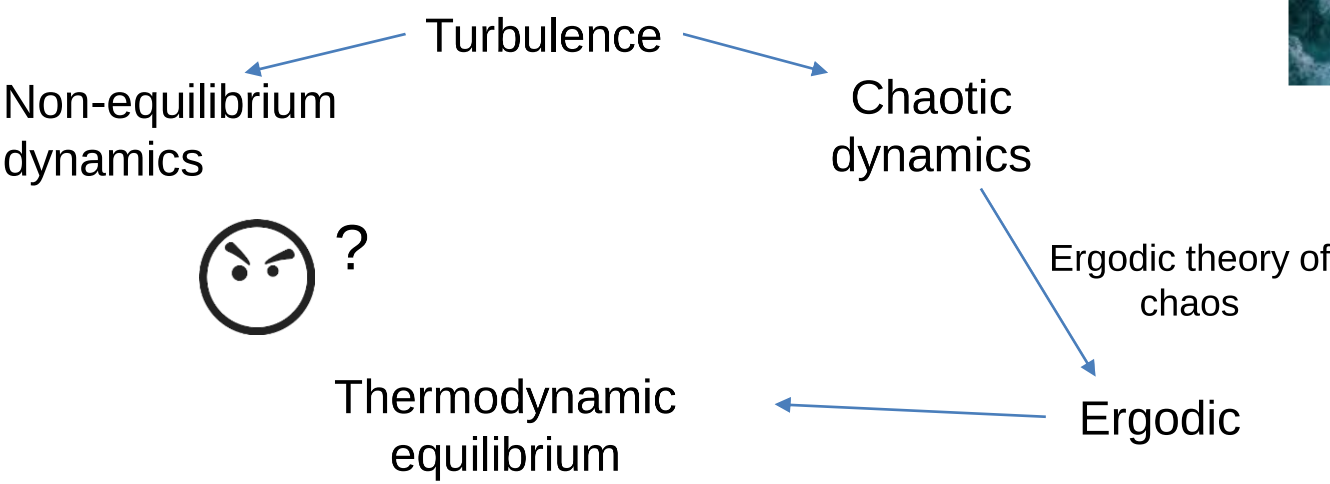




Chaotic Long-Range Order: Potential Origin

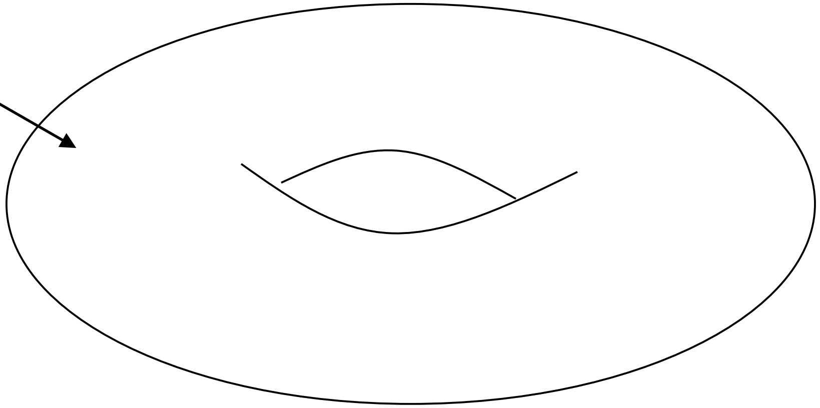
Chaotic Long-Range Order

Accidental picture	Symmetry breaking picture
CLRO is a "critical" phenomenon – some excitation has zero gap because the DS is at a "phase transition"	CLRO is a symmetry breaking phenomenon, and CLRO is a result of the Goldstone theorem.
Contradiction with ubiquity of CLRO	Requirement from ubiquity of CLRO
Self-Organized Criticality: postulation of existence of mysterious tendency of self-fine-tuning into the phase transition into ordinary chaos	All stochastic systems must possess such a symmetry and it must be a supersymmetry

ChT-SDE: Stochastic Evolution

$$\dot{x}(t) = F(x(t)) + (2\Theta)^{1/2} e_a(x(t))\xi^a(t)$$

DS's variables from some topological manifold, *X*, called phase space



ChT-SDE: Stochastic Evolution

Temperature

Flow vector field

$$\dot{x}(t) = F(x(t)) + (2\Theta)^{1/2} e_a(x(t))\xi^a(t)$$

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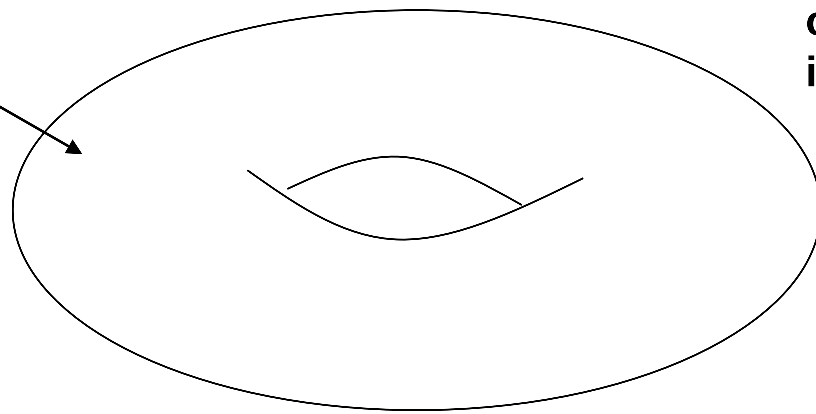
A set of vector fields, vielbeins, If (in)dependent on *x(t)*, it is said that noise is (additive)multiplicative



ChT-SDE: Stochastic Evolution

Temperature

Flow vector field

Guassian white Noise variables

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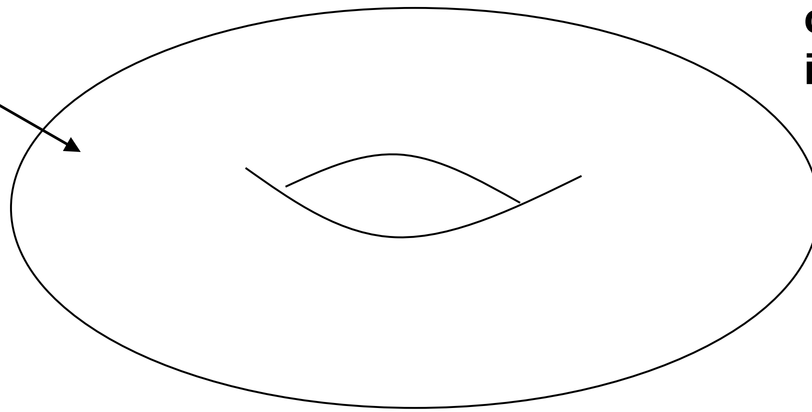

Possible generalizations: any noise; partial differential equations; flow vector field and vielbeins with explicit time/space dependence and "integral" or temporary (and spatially) nonlocal dependence on *x*…

ChT-SDE: Stochastic Evolution

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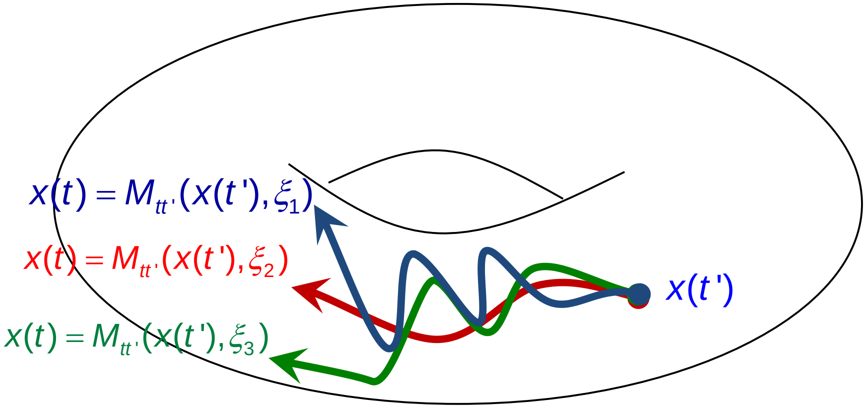
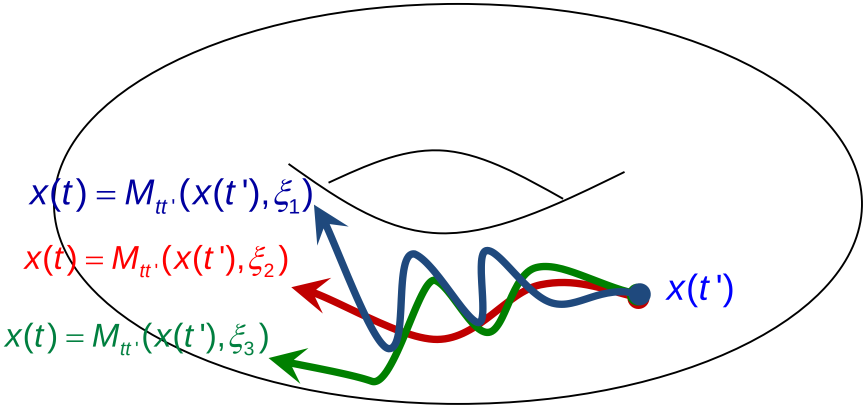


ChT-SDE: Stochastic Evolution

$$\dot{x}(t) = F(x(t)) + (2\Theta)^{1/2} e_a(x(t))\xi^a(t)$$

Noise dependent trajectories - family of maps

$$M_{tt'}(\xi) : x(t') \to x(t),\ x(t) = M_{tt'}(\xi, x(t'))$$



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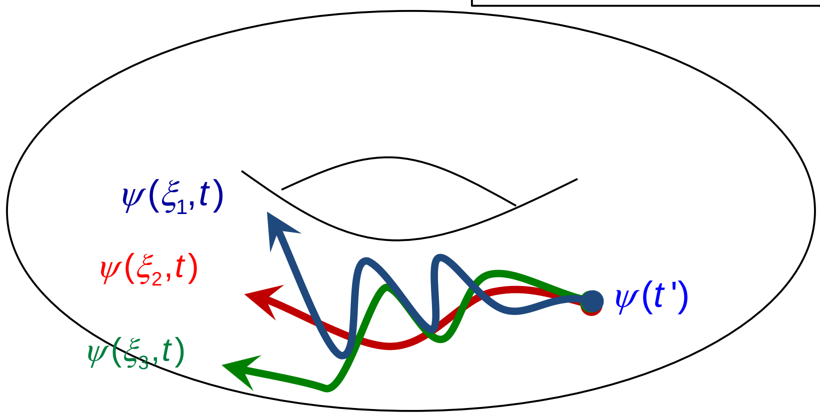


Time moment $t > t'$ — Temporal evolution — Time moment t'

$\psi(\xi, t)$ $\qquad$ $\psi(t')$

ChT-SDE: Hilbert Space

- GPDs in the coordinate-free setting = differential or k-forms:

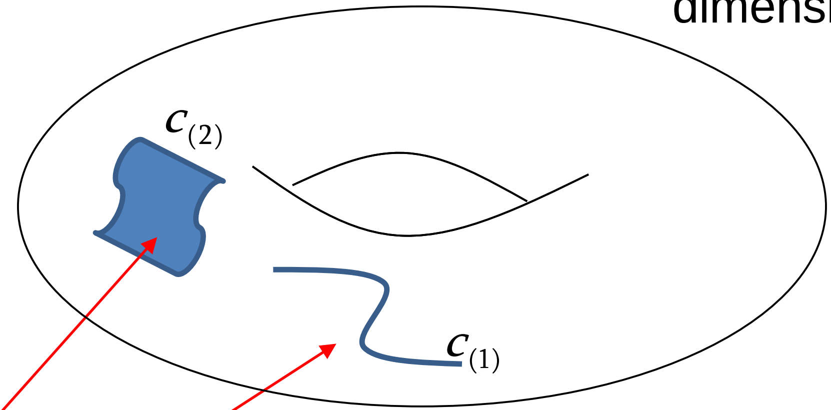
$$\psi^{(k)} = (k!)^{-1}\psi^{(k)}_{i_1\ldots i_k}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_k} \subset \Omega^{(k)}(X)$$

- Hilbert space is the exterior algebra $\Omega(X) = \bigoplus_{k=0}^{D}\Omega^{(k)}(X)$
- Consideration of generalized (not only total) probability distributions is a mathematical necessity

A k-form is naturally coupled to k-dimensional submanifolds (k-chains)

$$p^{(k)} = \int_{c_{(k)}} \psi^{(k)} \subset \mathbb{R}^1$$

<u>Meaning</u>: in local coordinates where k-chain belongs the hyperplane $x^{k+1},\ldots,x^{D} \to consts$ is the probability to find $x^1,\ldots,x^k$ within the chain, given the other variables are known



Examples: conditional and total probability distributions

$$p^{(2)}(x) = p(x)dx^1\wedge dx^2$$

$$p^{(1)}(x) = p(x^1\,|\,x^2)dx^1 + p(x^2\,|\,x^1)dx^2$$

Standard Conditional Probability Distribution

- GPDs in the coordinate-free setting = differential or k-forms:

$$\psi^{(k)} = (k!)^{-1} \psi^{(k)}_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k} \subset \Omega^{(k)}(X)$$

Example: standard definition of Cond.Prob.Density on $\mathbb{R}^D$

$$P_{tot}(x^1 \dots x^D) = P_{cond}(x^1 \dots x^k \mid x^{k+1} \dots x^D) P_{m\arg}(x^{k+1} \dots x^D)$$

In coordinate-free setting

$$P_{m\arg} = P_{m\arg}(x^{k+1} \dots x^D) dx^{k+1} \wedge \dots \wedge dx^D \subset \Omega^{D-k}$$

$$P_{cond} = P_{cond}(x^1 \dots x^k \mid x^{k+1} \dots x^D) dx^1 \wedge \dots \wedge dx^k \subset \Omega^k$$

$$P_{tot} = P_{cond} \wedge P_{m\arg} = P_{tot}(x^1 \dots x^D) dx^1 \wedge \dots \wedge dx^D \subset \Omega^D$$

ChT-SDE: Hilbert Space

Bra-ket “factorization” of total probability density

Quantum theory

$$\bar{\psi}(x)\psi(x) = \underbrace{TPF}_{\substack{Total\\ Probability\\ Function}}(x)$$

ChT-SDE

$$\bar{\psi}(x)\underbrace{dx^1 \ldots dx^k}_{\substack{\text{Unthermalized,}\\ \text{unstable vars.}}} \wedge \psi(x)\underbrace{dx^{k+1} \ldots dx^D}_{\substack{\text{Thermalized,}\\ \text{stable vars.}}} = TPF(x)\underbrace{dx^1 \ldots dx^D}_{\text{All vars.}}$$

ChT-SDE: Stochastic Evolution

$$\dot{x}(t) = F(x(t)) + (2\Theta)^{1/2} e_a(x(t))\xi^a(t)$$

Noise dependent trajectories - family of maps

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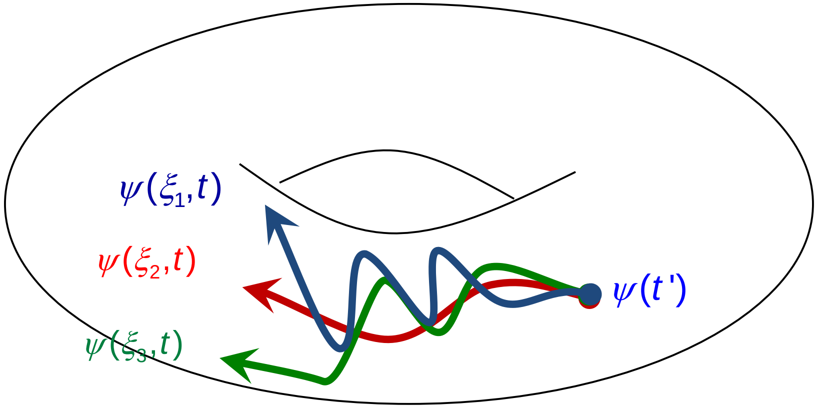

Time moment $t > t'$ — Temporal evolution ← — Time moment t'

$\psi(\xi,t)$ $\qquad$ $\psi(t')$

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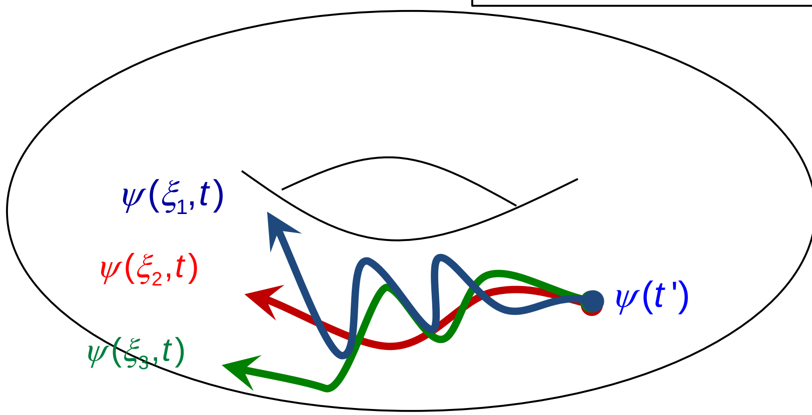

Time moment $t > t'$ — Temporal evolution ← — Time moment t'

$$\psi(\xi,t) = M^*_{t't}(\psi^{(k)}_{i_1\ldots i_k}(x(t'))dx^{i_1}(t) \wedge \ldots \wedge dx^{i_k}(t)$$

$$\psi(t') = \psi^{(k)}_{i_1\ldots i_k}(x(t'))dx^{i_1}(t') \wedge \ldots \wedge dx^{i_k}(t')$$

Pullback by $M_{t't}(\xi)$

Variable substitution

$$x(t') = M_{t't}(\xi, x(t))$$

ChT-SDE: Stochastic Evolution

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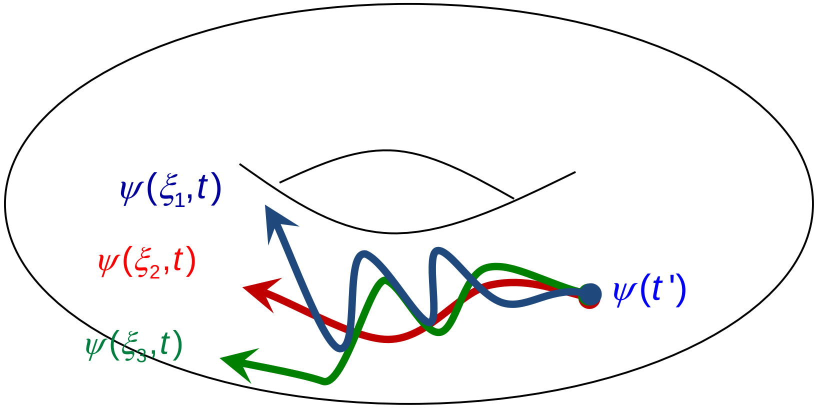

Stochastic evolution for GPDs, $\psi \subset \Omega(X)$, is

$$\psi(t) = \hat{M}_{tt'}\psi(t'),\ \ \hat{M}_{tt'} = \left\langle M^*_{t't}(\xi) \right\rangle_{Noise}$$

$M^*_{t't}(\xi)$ – pullback induced by $M_{t't}(\xi)$

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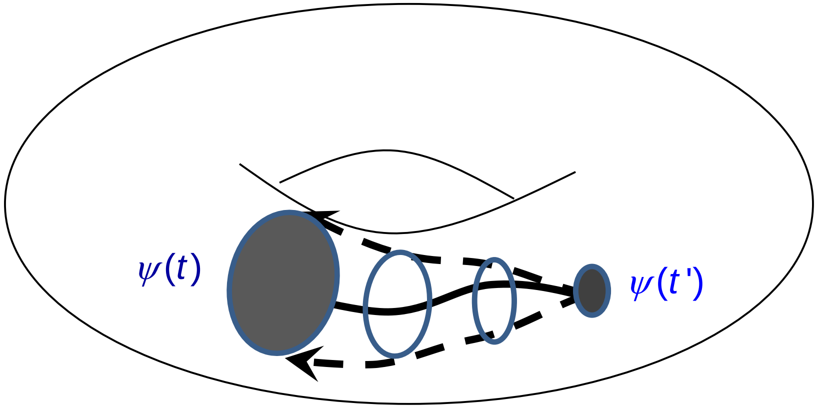

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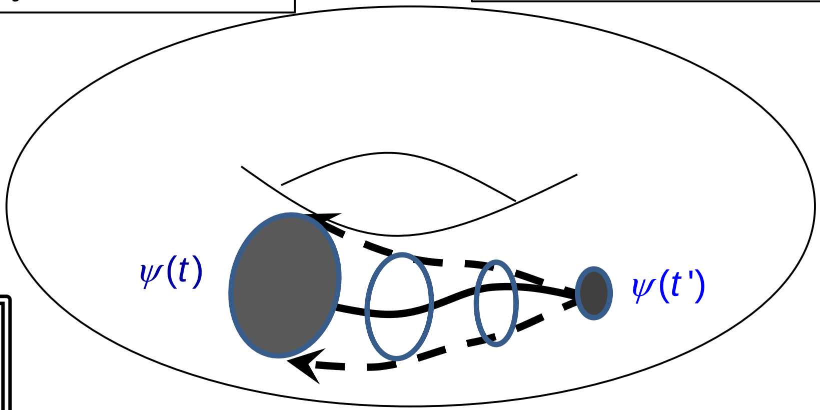
Noise dependent trajectories - family of maps

$M_{tt'}(\xi) : x(t') \to x(t),\ x(t) = M_{tt'}(\xi, x(t'))$

$$\hat{M}_{tt'} = e^{-(t-t')\hat{H}},\ \ \hat{H} = \hat{L}_F - \Theta\hat{L}_{e_a}\hat{L}_{e_a}$$

or infinitesemally, $\partial_t \psi = -\hat{H}\psi$



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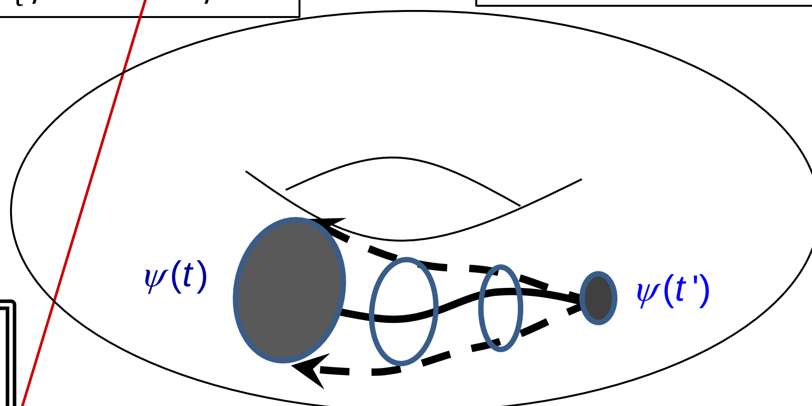

"Average" flow, Lie or physical derivative along F

Noise indiced diffusion

$$\hat{L}_{e_a}\hat{L}_{e_a} = g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \ldots$$

$g^{ij} = e^i_a e^j_a$ – noise induced metric

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$M^*_{t't}(\xi)$ – pullback induced by $M_{t't}(\xi)$

ChT-SDE: Stochastic Evolution

No Approximations!

$$\dot{x}(t) = F(x(t)) + (2\Theta)^{1/2} e_a(x(t))\xi^a(t)$$

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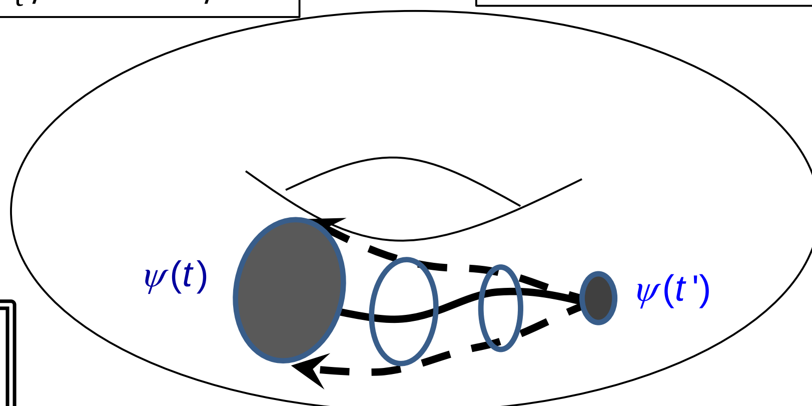

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$M^*_{t't}(\xi)$ – pullback induced by $M_{t't}(\xi)$

$$\partial_t P(x) d^D x = -(\partial_i F^i - \Theta \partial_i e^i_a \partial_j e^j_a) P(x) d^D x$$

Stratonovich Interp. = Weyl symmetrization

ChT-SDE: Supersymmetry

Stochastic evolution on exterior algebra

$$\partial_t \psi = -\hat{H}\psi, \ \hat{H} = \hat{L}_F - \Theta \hat{L}_{e_a} \hat{L}_{e_a}.$$

The Fokker-Planck operator can be given explicitely supersymmetric form

$$\hat{H} = [\hat{d}, \hat{\bar{d}}],$$

where $\hat{\bar{d}} = \hat{i}_F - \Theta \hat{i}_{e_a} \hat{L}_{e_a}$, $\hat{i}_F$ – interior multiplication, and the use of

Cartan formula, $\hat{L}_F = [\hat{d}, \hat{i}_F]$, has been made with $\hat{d}$ being exterior derivative.

Also,

$$[\hat{d}, \hat{H}] = 0,$$

$\hat{d}$ is a supersymmetry of the model

ChT-SDE: Meaning of Supersymmetry Operator

Exterior differentive:

$$\hat{d}\psi^{(k)}_{i_1\ldots i_k}(x)dx^{i_1}\wedge\ldots\wedge dx^{i_k} = \frac{\partial}{\partial x^j}\psi^{(k)}_{i_1\ldots i_k}(x)dx^j\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}$$

is very fundamental to albegraic topology

-) It is the matter of Stokes' theorem, $\int_{\partial c_{k+1}}\psi^{(k)} = \int_{c_{k+1}}\hat{d}\psi^{(k)}$,

where ∂ is the boundary operator.

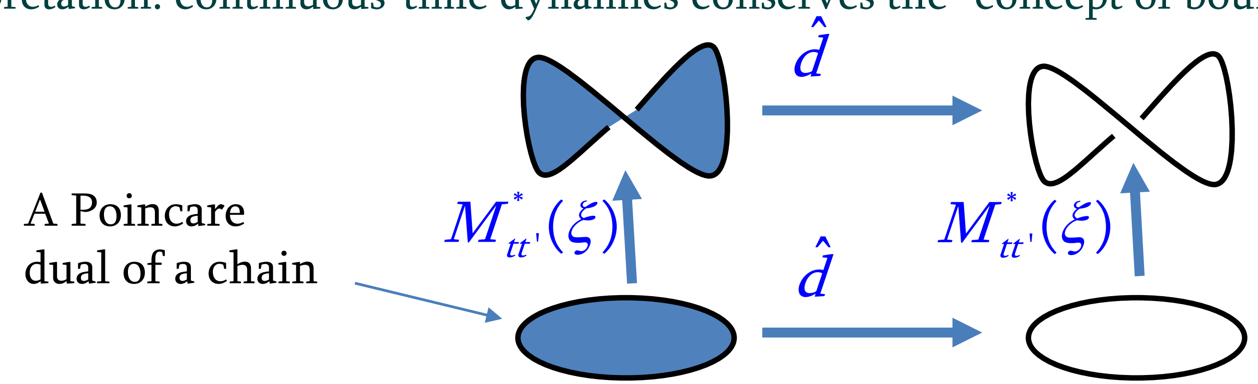
-) Its cohomology is De Rahm cohomology

$\hat{d}$ is the algebraic representative of "boundary" operator

It is nilpotent, $\hat{d}^2 = 0$, "boundary of a boundary" is empty

$\hat{d}$ commutes with any pullback $[\hat{d}, M^*_{tt'}(\xi)] = 0$, and thus with the evolution operator $[\hat{d}, \hat{M}] = 0$

Interpretation: continuous-time dynamics conserves the "concept of boundary"



ChT-SDE: Meaning of Supersymmetry Operator

Properties of wedge product

$$dx^{i_1} \wedge dx^{i_2} = -dx^{i_2} \wedge dx^{i_1}$$

are those for anticommuting or fermionic fields

$$\chi^{i_1} \chi^{i_2} = -\chi^{i_2} \chi^{i_1}$$

Differential forms can be given as functions

of bosonic and fermionic fields

$$\psi^{(k)}_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} = \psi^{(k)}_{i_1 \dots i_k} \chi^{i_1} \dots \chi^{i_k}$$

In these terms, exterior derivative has the form

$$\hat{d} = \chi^{i_1} \frac{\partial}{\partial x^i}$$

ChT-SDE: Supersymmetry

Stochastic evolution on exterior algebra

$$\partial_t \psi = -\hat{H}\psi, \; \hat{H} = \hat{L}_F - \Theta \hat{L}_{e_a} \hat{L}_{e_a}.$$

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Also,

$$[\hat{d}, \hat{H}] = 0,$$

$\hat{d}$ is a supersymmetry of the model

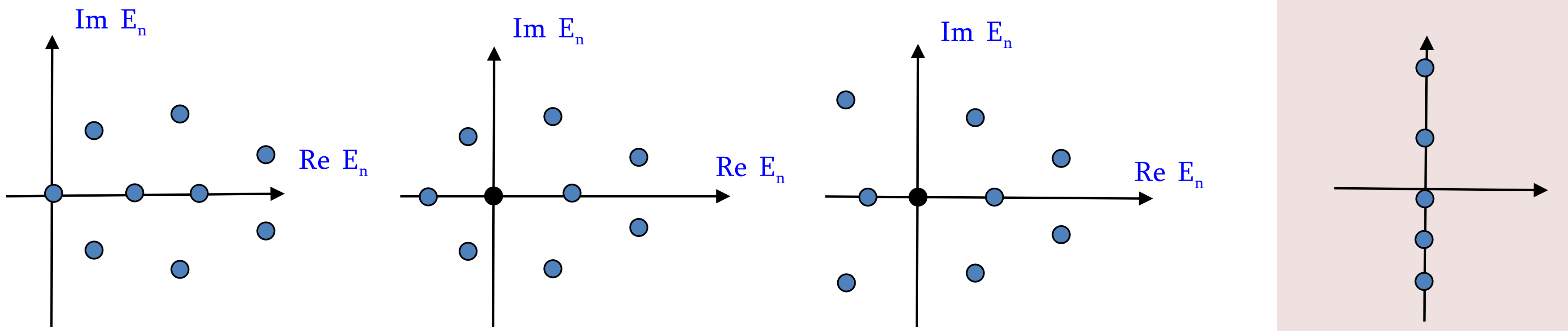
Eigenstates of $\hat{H}$ are either

-) supersymmetric singlets: $\hat{d}|\theta_n\rangle = 0$ but $|\theta_n\rangle \neq \hat{d}|\text{something}\rangle$

All have zero eigenvalues !

-) or non-supersymmetric doublets: $|\vartheta_n\rangle$ and $\hat{d}|\vartheta_n\rangle \neq 0$

ChT-SDE: Possible FP Spectra



d-symmetry broken

d- and PT reversal symmetries broken

Quantum theory
$\partial_t \psi = -(i\hat{H})\psi$

FP operator is real and thus psuedo-Hermitian

- Eigenvalues are either real or complex conjugate pairs (Ruelle-Pollicott resonances of DS theory)
- Eigensystem is complete, bi-orthogonal: $\hat{H}|n\rangle = E_n|n\rangle, \langle n|\hat{H} = E_n\langle n|, \sum_n |n\rangle\langle n| = 1, \langle k|n\rangle = \delta_{kn}$

For physical models with positive definite noise-metric

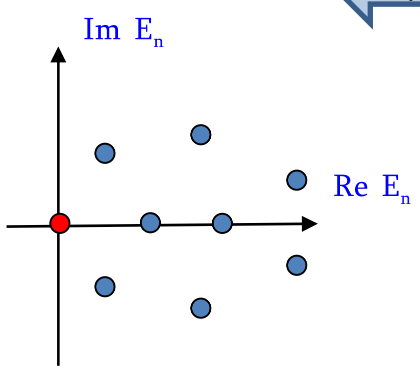
- Real parts of eigenvalues are bounded from below

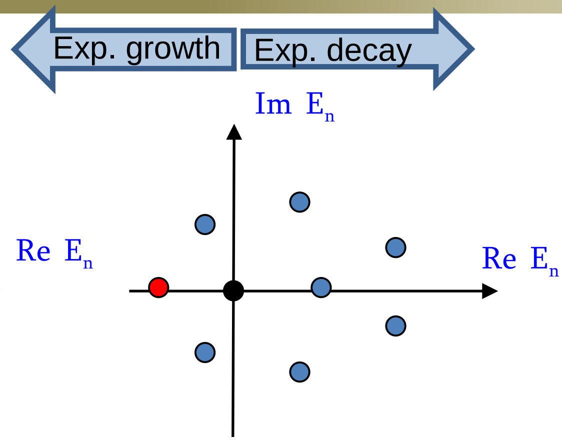
From supersymmetry

- All eigenstates are either supersymmetric singlets or non-supersymmetric doublets
- All non-zero eigenvalues correspond to non-supersymmetric pairs
- There always exist a supersymmetric state of the steady-state total probability distribution,

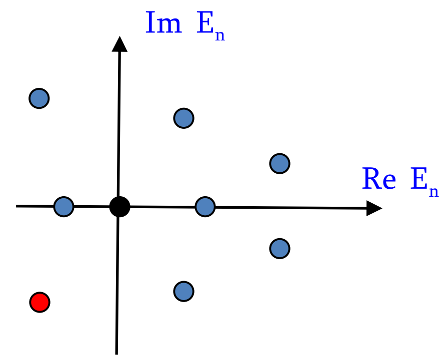
i.e., the state of "thermodynamic equilibrium".

ChT-SDE: Possible FP Spectra





d-symmetry broken



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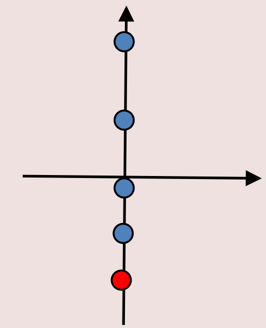


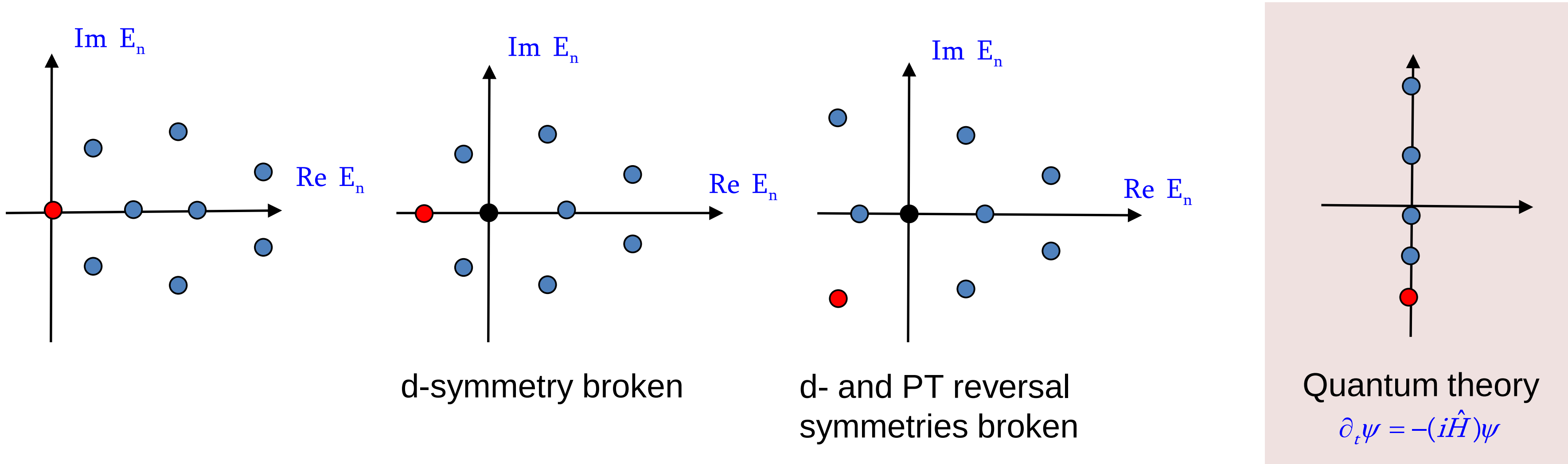
Quantum theory

$\partial_t \psi = -(i\hat{H})\psi$

	Pathintegral representation	Contribution only from	Physical meaning	Value
Partition Function $Z = Tre^{-t\hat{H}}$	$\iint_{APBC} D\Phi e^{\{Q,\Psi\}}$	Ground states (in large time limit)	Stochastic number of periodic solutions (for some models)	$Z \xrightarrow{t\to\infty} 2e^{t\|E_g\|}$ For spectrum b Chaotic behavior
Witten Index $W = Tr(-1)^{\hat{F}} e^{-t\hat{H}}$	$\iint_{PBC} D\Phi e^{\{Q,\Psi\}}$	Supersymmetric states only	Partition function of noise (up to a topological factor)	Euler characteristic of X

Both have physical meaning only if the entire exterior algebra is the Hilbert space !

ChT-SDE: Ground States and Ergodicity



Spectra a and b: correlators in the long time limit reduce to those of the ground states only

$$\left\langle \hat{O}_1(t_1)...\hat{O}_k(t_k)\right\rangle = Z^{-1}_{t_{+\infty}t_{-\infty}} Tr e^{-(t_{+\infty}-t_1)\hat{H}} \hat{O}_1 e^{-(t_1-t_2)\hat{H}} ... e^{-(t_{k-1}-t_k)\hat{H}} \hat{O}_k e^{-(t_k-t_{-\infty})\hat{H}}$$

$$\xrightarrow{t_{\pm\infty}\to\pm\infty} N_g^{-1} \sum\nolimits_g \langle g | \hat{O}_1(t_1)...\hat{O}_k(t_k) | g \rangle,$$

where $\hat{O}_1(t_1) = e^{-(t_{-\infty}-t_1)\hat{H}} \hat{O}_1 e^{-(t_1-t_{-\infty})\hat{H}}$ in Heisenberg picture

Interpretation: ergodicity property

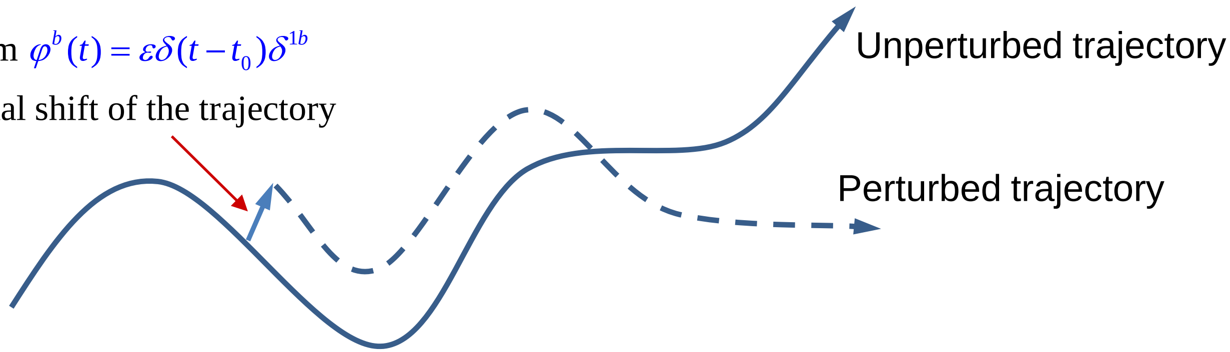
For spectra c: one can use standard trick of quantum theory

(a little Wick rotation) to make theory "ergodic"

ChT-SDE: Butterfly Effect

For perturbation of the form $\varphi^b(t) = \varepsilon\delta(t - t_0)\delta^{1b}$

$\varepsilon f_1(x(t_0))$ – the infinitesemal shift of the trajectory



In order to study response of the system, introduce probing fields

$F^i(x(t)) \to F^i(x(t)) + \phi^b(t) f_b^i(x(t))$ ⟵ External perturbation

The evolution operator transforms as

$$\hat{H} \to \hat{H}(t) = \hat{H} + \phi^b(t)\hat{L}_{f_b} = \hat{H} + \phi^b(t)\left[\hat{d}, \hat{\imath}_{f_b}\right],$$

The response can be charactized by response correlators ($Z_t = Tr \underset{\text{Chron.Ordering}}{\mathfrak{T}} e^{-\int_0^t \hat{H}(\tau)d\tau}$)

$$\underset{t\to\infty}{Lim} Z_t^{-1} \left.\frac{\delta Z_t(\phi)}{\delta\phi^{b_1}(t_1)....\delta\phi^{b_k}(t_k)}\right|_{\phi\to 0} = N_g^{-1} \sum_{g\subset\text{ground states}} \langle g | \mathfrak{T} \underbrace{\left[\hat{d}, \hat{\imath}_{f_{b_1}}(t_1)\right] ... \left[\hat{d}, \hat{\imath}_{f_{b_k}}(t_k)\right]}_{\text{In Heisenberg picture}} | g \rangle$$

$$= N_g^{-1} \sum_{g\subset\text{ground states}} \langle g | [\hat{d}, \hat{A}] | g \rangle, \text{ where } \hat{A} = \mathfrak{T}\hat{\imath}_{f_{b_1}}(t_1) ... \left[\hat{d}, \hat{\imath}_{f_{b_k}}(t_k)\right]$$

All vanish by the definition of supersymmetric (ground) states
Interpretation: forgets perturbations/initial conditions in the long-time limit

Some do not vanish because the ground states are not supersymmetric
Interpretation: remembers perturbations/initial conditions even in the infinitely long time limit – the butterfly effect !

ChT-SDE: Expectation Values and Statistics

Expectation value of an operator, $\hat{O}$, at moment t,

$$\left\langle \hat{O}(t) \right\rangle = \lim_{T\to\infty} Z_T^{-1} \sum\nolimits_n \langle n | e^{-(T-t)\hat{H}} \hat{O} e^{-t\hat{H}} | n \rangle = N_g^{-1} \sum\nolimits_g \langle g | \hat{O} | g \rangle$$

For wide class of operators including $\hat{O} = O \subset \Omega^0$

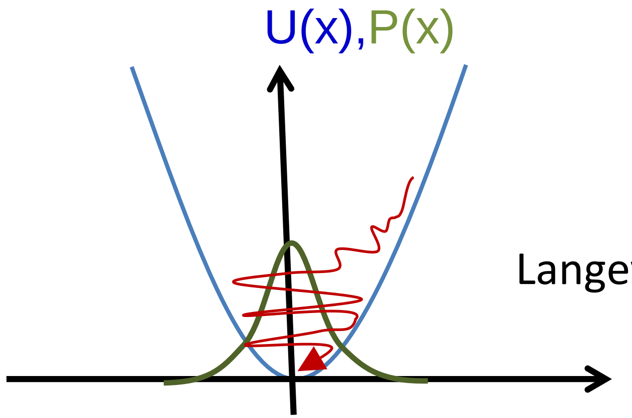
$$\left\langle \hat{O} \right\rangle = \int_X O(x) P_g(x)$$

where the "ergodic" probability density

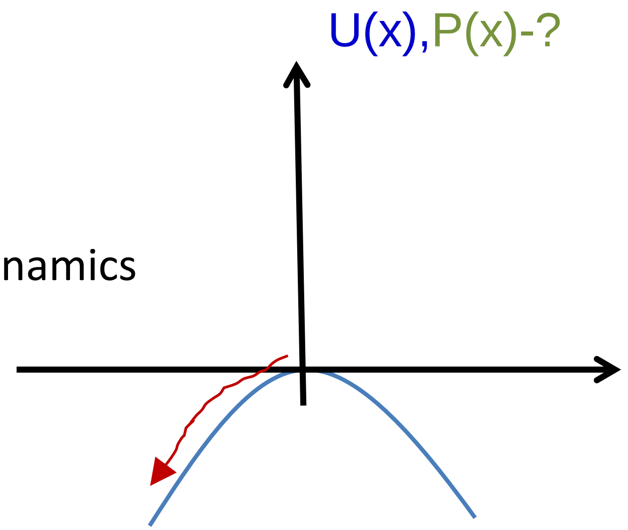
$$P_g(x) = N_g^{-1} \sum\nolimits_g \bar{\psi}_g(x) \psi_g(x) dx^1 ... dx^D$$

Statistics works no matter if the supersymmetry is broken or not

ChT-SDE: Deterministic Ground States



Langevin 1D dynamics



- Forgets initial conditions

- Thermalizes to a steady-state total probability distribution (the ground state)

$$\langle g| = 1 \qquad |g\rangle = P(x)dx$$

$$\langle g|g\rangle = \int P(x)dx = 1$$

- Never forgets initial conditions because the variable is not stable

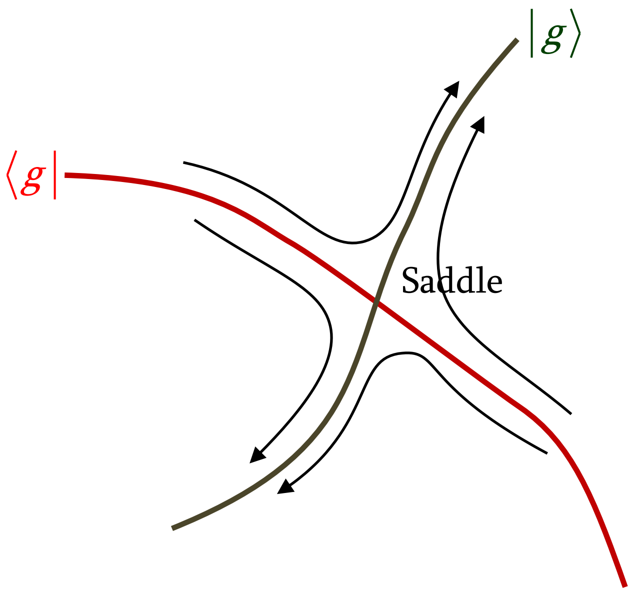
- The steady-state total probability distribution is meaningless . The ground state is

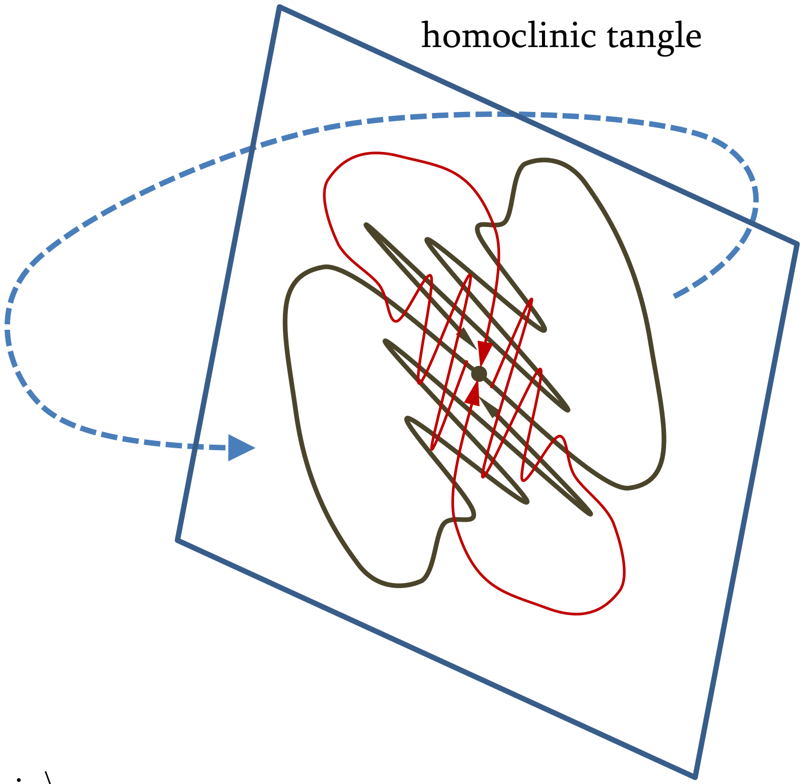
$$\langle g| = P(x)dx \qquad |g\rangle = 1$$

$$\langle g|g\rangle = \int P(x)dx = 1$$

The ground state is <u>not</u> a distribution in unstable variables

ChT-SDE: Deterministic Ground States





Ket – Poincare dual of global unstable manifolds
Bra- Poincare dual of global stable manifolds

$$\hat{d}\,|\text{Poincare dual of chain}\rangle = |\text{Poincare dual of boundary of chain}\rangle$$

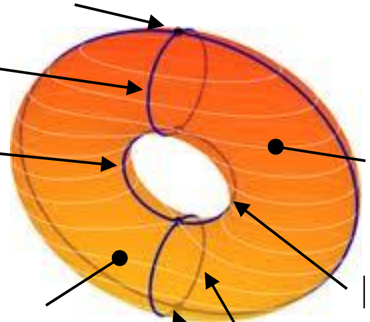
Integrable (non-chaotic) flows: the supersymmetric ground states are Poincare duals of global unstable manifolds

Non-integrable (chaotic) flows: the non-supersymmetric ground states are Poincare duals of global unstable manifolds modified by a functional dependence on position

ChT-SDE: Deterministic Ground States

Langevin ODE on 2D torus

Langevin function is the height in the "3rd" direction

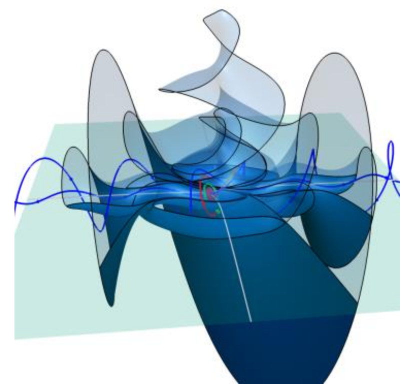


$$W = (-1)^0 \langle 3|3\rangle + (-1)^1 \langle 2|2\rangle + (-1)^1 \langle 1|1\rangle + (-1)^2 \langle 0|0\rangle = 0$$ (Euler characteristic of torus)

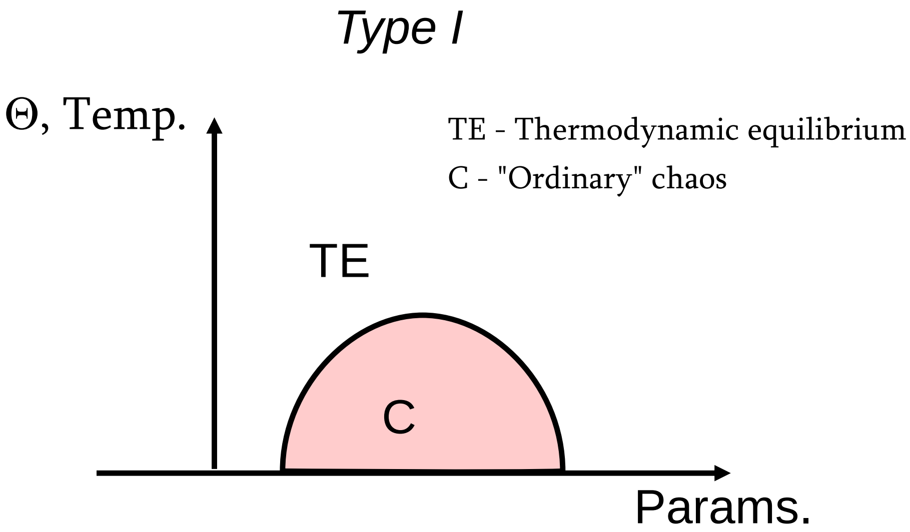
Lorenz model

Unstable manifold

(picture by Hinke Osinga)



ChT-SDE: Stochastic Phase Diagram



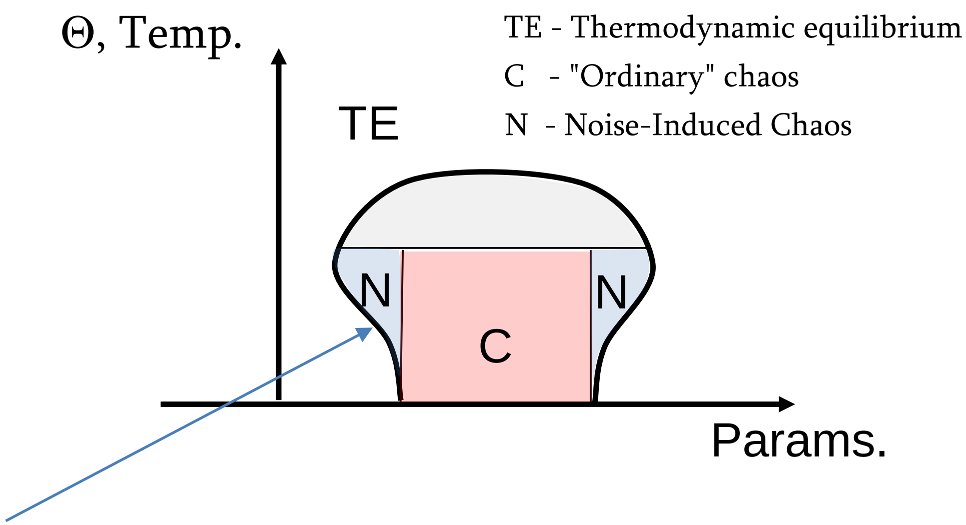
High temp.: $\hat{H} = \hat{L}_F - \Theta \hat{L}_{e_a} \hat{L}_{e_a} \xrightarrow{\Theta\to\infty} -\Theta \hat{L}_{e_a} \hat{L}_{e_a}$

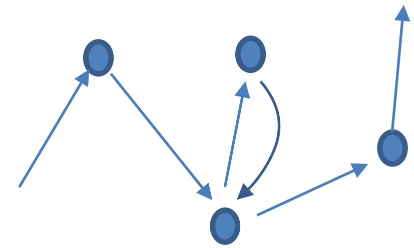
Physical Laplacians do not break supersymmetry.

With the increase of noise temperature, the susy will eventually get restored – strong enough noise destroys chaotic long-range order

ChT-SDE: Stochastic Phase Diagram

Type II





Noise-induced Chaos

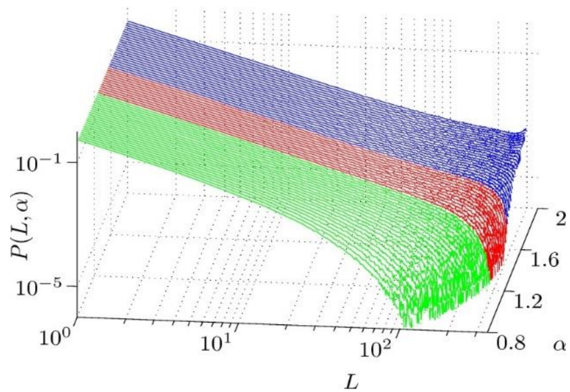
- Weak noise introduces exponentially weak overlap or noise-induced tunneling processes (instantons) between “local” ground states on different attractors.
- When tunneling processes break susy, Goldstinos representing parameters (modulii) are gapless. Result: power-law or scale free statistics.
- Mostly “regular” behavior along attractors interrupted by sudden unpredictable processes with scale-free statistics. Exists on the border of “ordinary” chaos. Typical description of Self-Organized Criticality!
- At higher temperatures, N-C is a crossover because external observer can not tell between different tunneling events

Example: Neurodynamics

Experimental evidence:
neuroavalanches exhibit power-law statistics.
This suggests brain has its susy broken by these processes – the N-phase

Numerical Evidence:
Neuroavalanches exhibit power-law for a range of parameters.

A. Levina, J.M. Herrmann, and T. Geisel (2006). Dynamical Synapses Give Rise to a Power-Law Distribution of Neuronal Avalanches



Neuronal Avalanches in Neocortical Circuits

John M. Beggs and Dietmar Plenz
Unit of Neural Network Physiology, Laboratory of Systems Neuroscience, National Institute of Mental Health, Bethesda, Maryland 20892

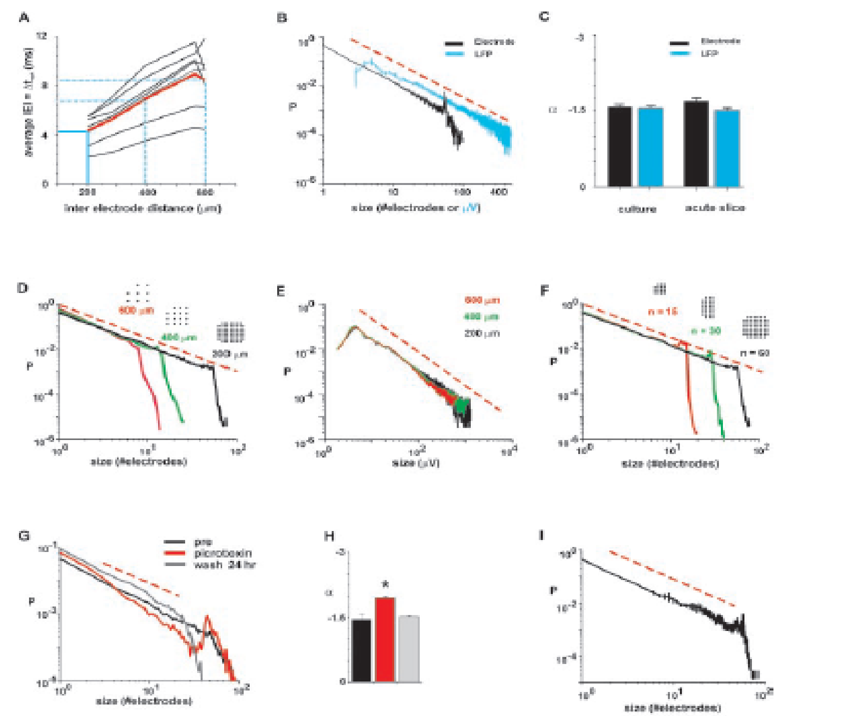


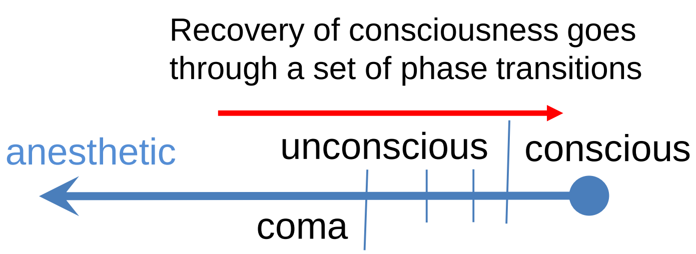
Figure 4. Characteristic exponent for neuronal avalanche sizes is $-3/2$. *A*, IED versus IEI_{avg} for original and rescaled grid sizes. Red, Average; black, individual cultures. *B*, Power laws at $\Delta t = IEI_{avg}$ for each culture have characteristic exponent $\alpha \sim -1.5$. Black, Number of electrodes; blue, LFP; average for all cultures. *C*, Average slopes for cultures (left) and acute slices (right). *D*, At $\Delta t = IEI_{avg}$ and corresponding IED, the slope α is independent of array size. Icons indicate resampled arrays at IED = 200, 400, and 600 μm. *E*, Resampled power laws for summed LFP values (same arrays as in *D*). *F*, Cutoff point of the power law is determined by the number of electrodes in the array (n = 15, 30, 60; IED = 200 μm). *G*, Reduction in inhibition in the presence of the $GABA_A$ receptor antagonist picrotoxin destroys the power law and renders the event size distribution bimodal. Note the presence of a large hump at higher values, indicating epileptic discharge. *H*, The initial slope of the event size distribution is significantly steeper ($p < 0.05$) in the presence of picrotoxin. Same color code as in *G*. *I*, Average event size distribution for refractory period set to 0 msec at Δt = 4 msec (three cultures). Broken line in red indicates slope of $-3/2$.

Example: Neurodynamics

Recovery of consciousness is mediated by a network of discrete metastable activity states

Andrew E. Hudson[a,1], Diany Paola Calderon[b,1], Donald W. Pfaff[b,2], and Alex Proekt[b,c,2]

[a]Department of Anesthesiology and Perioperative Medicine, David Geffen School of Medicine, University of California, Los Angeles, CA 90095; [b]Laboratory for Neurobiology and Behavior, The Rockefeller University, New York, NY 10065; and [c]Department of Anesthesiology, Weill Cornell Medical College, New York, NY 10021



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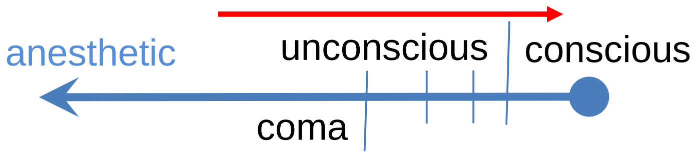
Recovery of consciousness goes through a set of phase transitions



Atkinson and Shiffrin model: short-term memory and long-term memory

Scale:~<1min
Physics: fast electrochemical dynamics of light ions
SDEs: well know, e.g., Hudgkin-Huxley model

Scale:~>1min
Physics: bio-chemical and mechanical rewiring of synapses

Example: Neurodynamics

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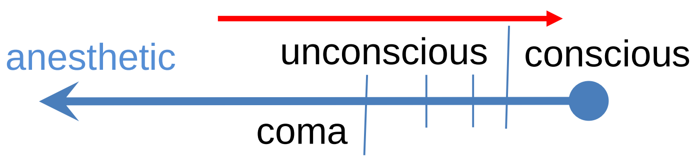
Determined by the same neurodynamical phenomena

Recovery of consciousness goes through a set of phase transitions



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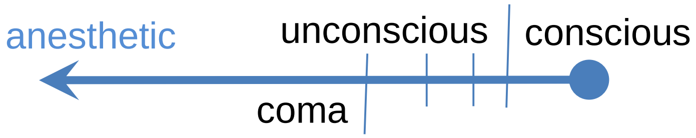
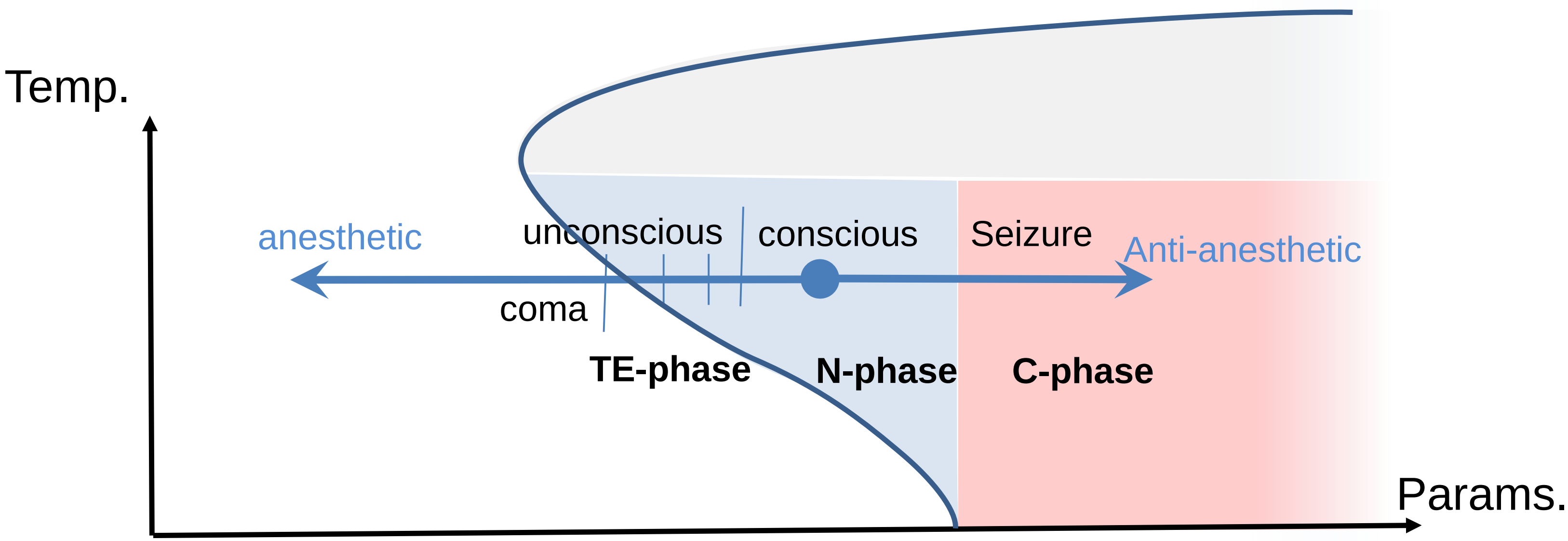
The ChT-SDE picture



The ChT-SDE picture



-) Conscious brain is in the N-phase. One can tell between neuroavalanches-this is weak-noise regime.
-) C-phase must correspond to neurodynamical phenomenon of "seizure" (like in epilepsy) – neurons fire non-stop, non-integrable flow. Again, because the N-C transition is sharp, the brain is in the weak-noise regime
-) Coma is in the TE-phase where there is no chaotic dynamical memory. Therefore, chaotic dynamical memory is the short-term neurodynamical memory.
-) Existence of short-term memory is a necessary condition for being conscious. Consciousness is within the N-phase

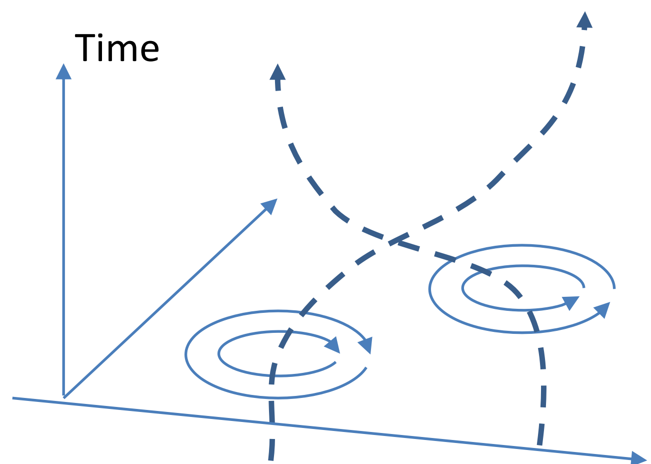
ChT-SDE: Low-Energy Effective Theory

Model	Symmetry broken	Order Parameter	Goldstone-Nambu Particle	Low-Energy Effective Theory
Ferromagnet	O(3)	Local Magnetization	Spinwave	LLG Equation
Crystal	Spatial translation	Local Stress	Transverse sound	Theory of Elasticity
Supersonductor	(Global) U(1)	“Wavefunction” of Bose-Condensate of Cooper pairs	Zero sound	GL Theory
…		…		…
Chaotic DS	Top. Susy	Unstable variables	Goldstinos	?

Example: 2D vortex “dominated” turbulence

- Order parameter: (half-of) spatial positions of vortices

- Goldtinos are supersymmetric partners of position of vortices
- LEET: could it be Schwartz-type TFT? If yes, does the concepts of braiding, topological quantum computing etc. apply somehow?



Conclusion

The newly found approximation-free theory of stochastic dynamics

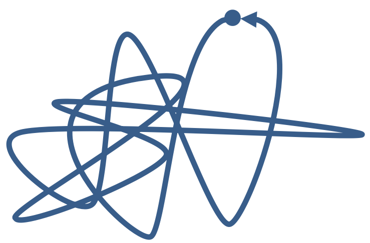
- reveals the mathematical origin of the ubiquitous dynamical/chaotic long-range – the spontaneous breakdown of topological supersymmetry that all natural/stochastic dynamical systems possess. “Chaos” (absence of order) is a misnomer in a certain sense because dynamical chaos is a low-symmetry or “ordered” phase.
- clarifies the concepts of thermodynamic equilibrium and ergodicity
- demystifies the controversial concept of self-organized criticality
- shows that dynamical properties of chaotic DSs can not be described by statistics. Low-energy effective theories of chaotic models, such as turbulent water, are those of gapless goldstinos/fermions – supersymmetric partners of unthermalized variables
- because of its multidisciplinary character and widest applicability, has a potential to bring together specialists in DSs theory, statistics, and topological field theories and/or algebraic topology that will result in cross fertilization of these mathematical disciplines
- has a potential to bring the studies of non-quantum systems to the new level of mathematical rigor and beauty, i.e., the level of supertsymmetric quantum (field) theories

Thank You!

ChT-SDE: Transient Processes

Ergodic Dynamics

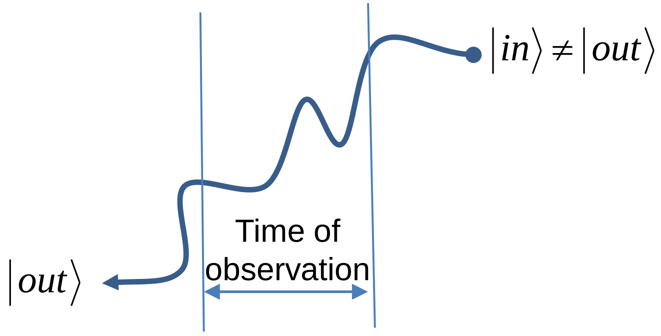
Infinitely long dynamics with the global ground state. Periodic B.C.



Examples: turbulent water, brain …

Transients (weak noise)

Finite-time dynamics starting and ending at different points; composite instanton

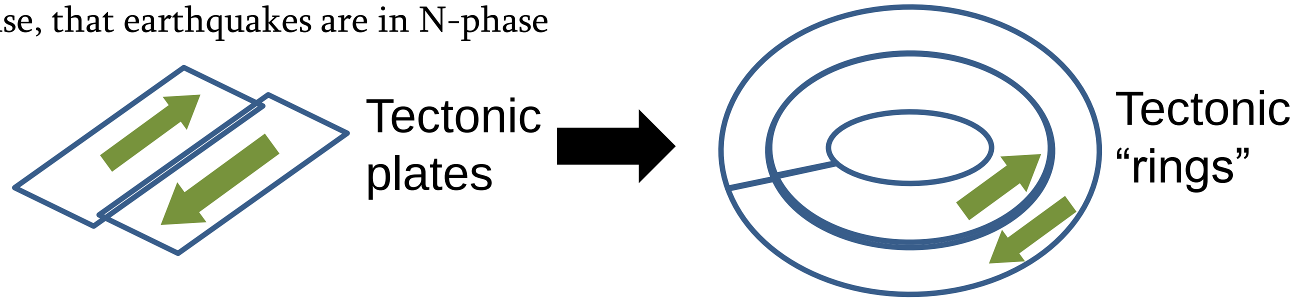


Examples: crumbling paper, Barkhausen jumps in ferromagnets, glasses, cascades and chain reactions of various types …

d-symmetry is intrinsically broken on instantons – crackles must (and they do) exhibit power-law statistics

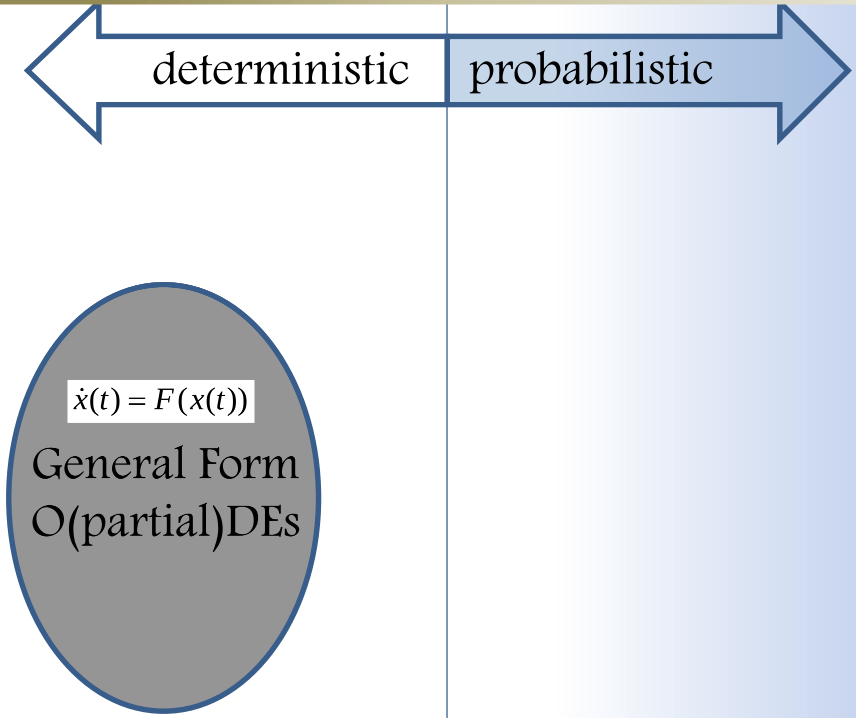
Transients can be though of being in the N-phase only in a sense of equivalent "ergodic" theory.

It is in this sense, that earthquakes are in N-phase



Additional Slides

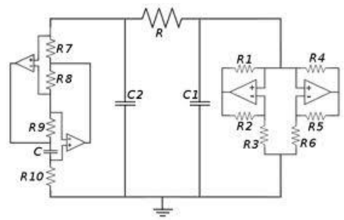
Where to Search?



Where to Search?

$$\begin{cases} \dot{x} = \partial H / \partial p \\ \dot{p} = -\partial H / \partial x \end{cases}$$

$$C_1 \frac{dv_1}{dt} = \frac{(v_2 - v_1)}{R} - h(v_1)\,,$$
$$C_2 \frac{dv_2}{dt} = \frac{(v_1 - v_2)}{R} + i_L\,,$$
$$L \frac{di_L}{dt} = -v_2\,,$$



-) Hamilton/Classical, Conservative dynamics

-) Electric circuitry

-) Magnetodynamics (LLG)

$$\frac{\partial \mathbf{m}}{\partial t} = -\gamma\, \mathbf{m} \times \mathbf{H}_{\text{eff}} + \alpha\, \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t}\,.$$

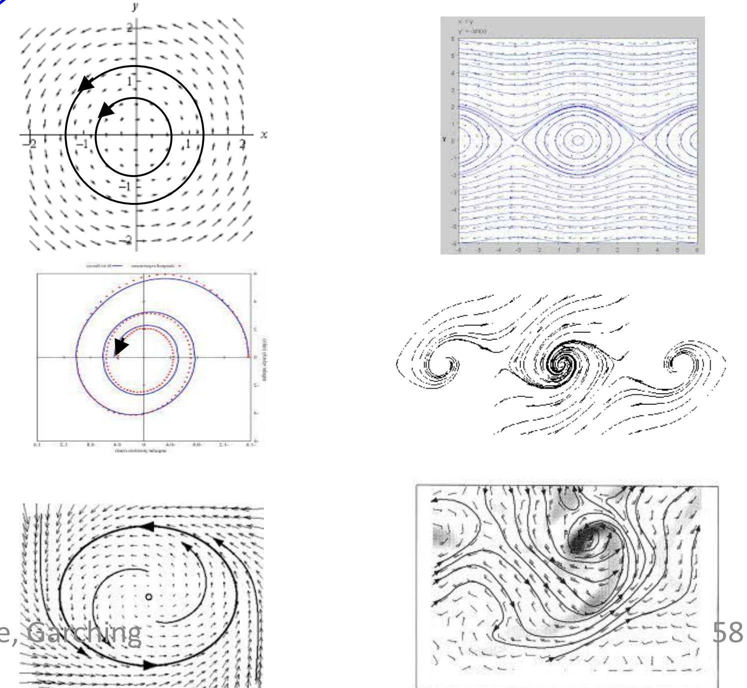
-) Schrodinger equation

$$i\hbar \dot{\psi} = (-\hat{p}^2 / 2m + V)\psi$$

-) (Magneto-)Hydro-, Bio-, Econo-, Neuro- Dynamics

Everything in Nature !

Flow vector field: Phase Portrait



System's variables, points from a topological manifold called the phase space.

$$\dot{x}(t) = F(x(t))$$

Where to Search?

$$S = \int (p\dot{x} - H(xp))dt \quad \begin{cases} \dot{x} = \partial H / \partial p \\ \dot{p} = -\partial H / \partial x \end{cases}$$

$$C_1 \frac{dv_1}{dt} = \frac{(v_2 - v_1)}{R} - h(v_1)\,,$$
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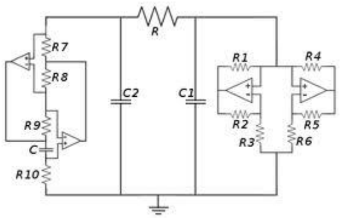

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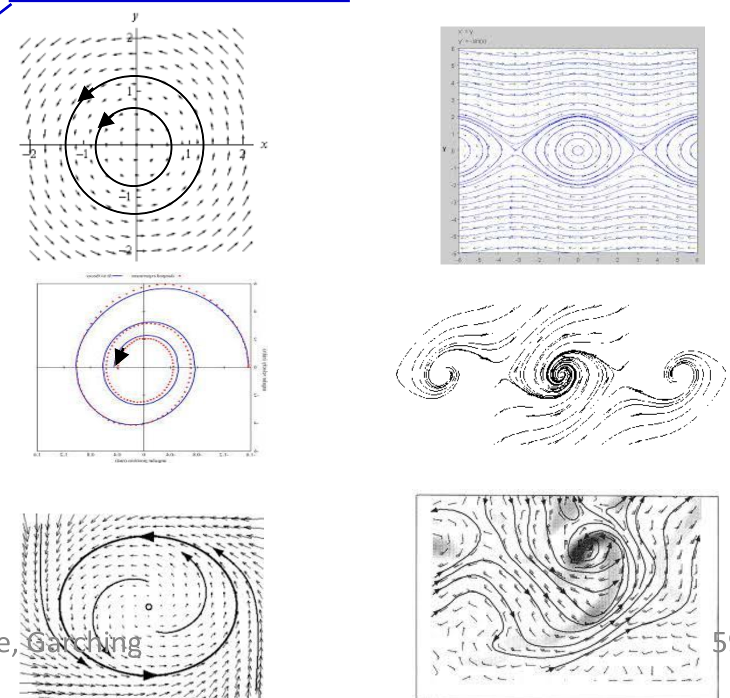
$$i\hbar\dot{\psi} = (-\hat{p}^2 / 2m + V(x))\psi \qquad S = \int \overline{\psi}(i\hbar\partial_t - (\hat{p}^2 / 2m + V(x))\psi dt d^d x$$

Everything in Nature !

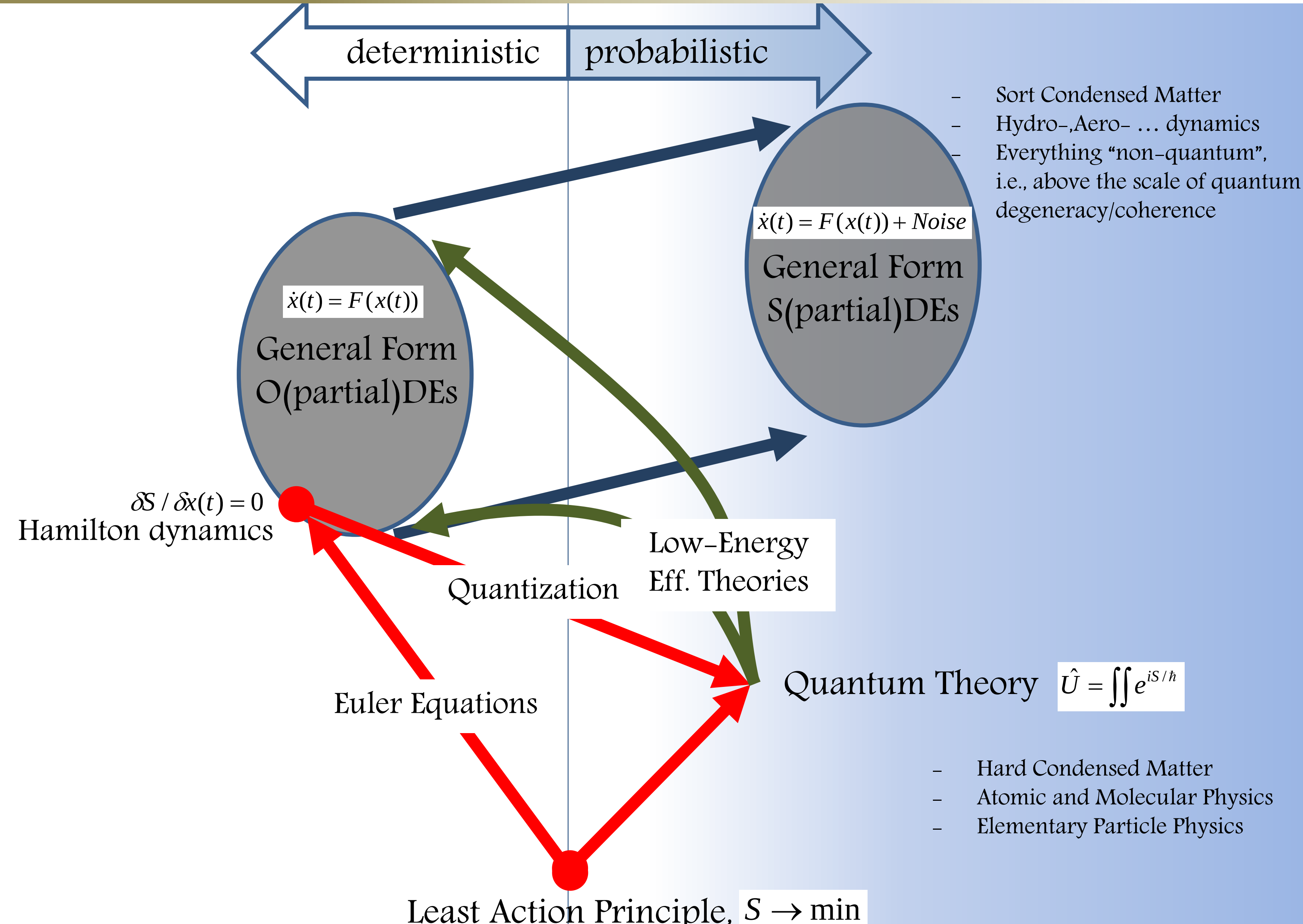
System's variables, points from a topological manifold called the phase space.

$$\dot{x}(t) = F(x(t))$$

Flow vector field: Phase Portrait



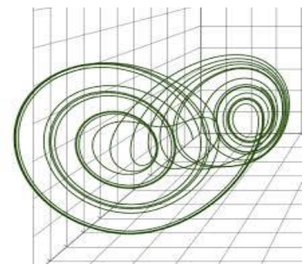
Where to Search?



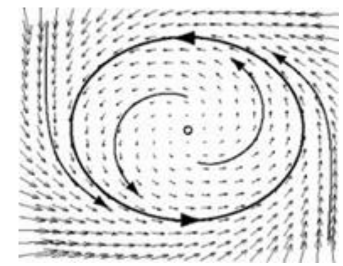
Where to Search?

ODE is the domain of the DSs Theory

Deterministic Chaos is the major discovery within DS theory
-) "the three-body problem" by Poincare (1887)
-) Numerical rediscovery by Lorenz (1963) and others.



Vs.



In hydrodynamics, chaos is known as turbulence

Deterministic chaos in ODEs– non-integrability in the sense of DS theory

This definition does not lead to the explanation of the Butterfly Effect

Where to Search?

SDE = ODE + Noise

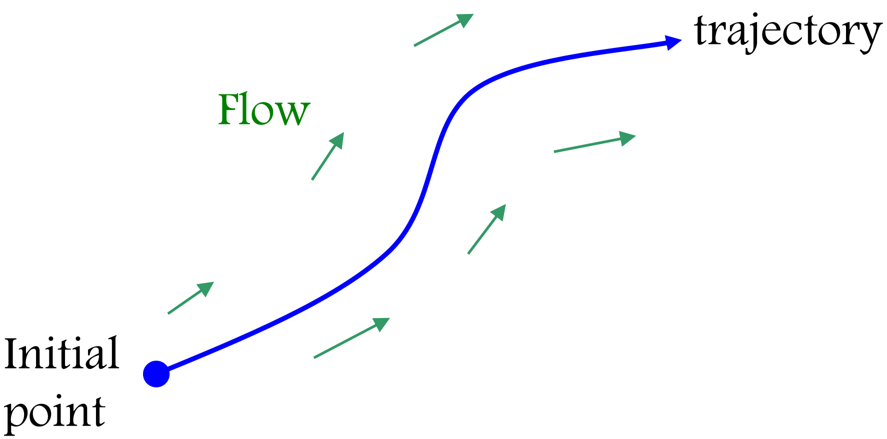
$$\partial_t x(t) = F(x,t) + e_a(x)\sqrt{2\Theta}\,\xi^a(t)$$

noise

A set of vector fields

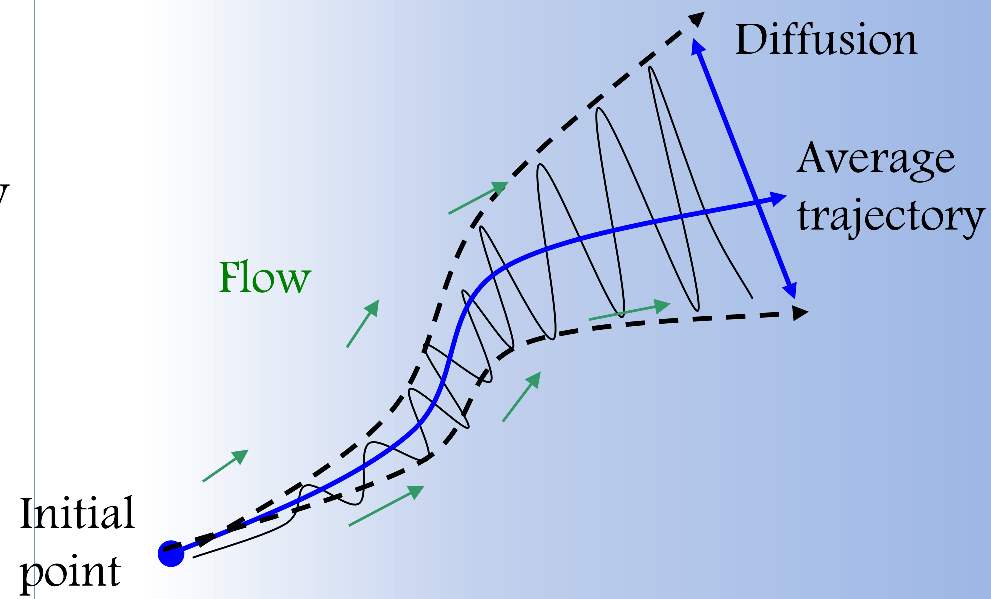
temperature

ODE



Deterministic

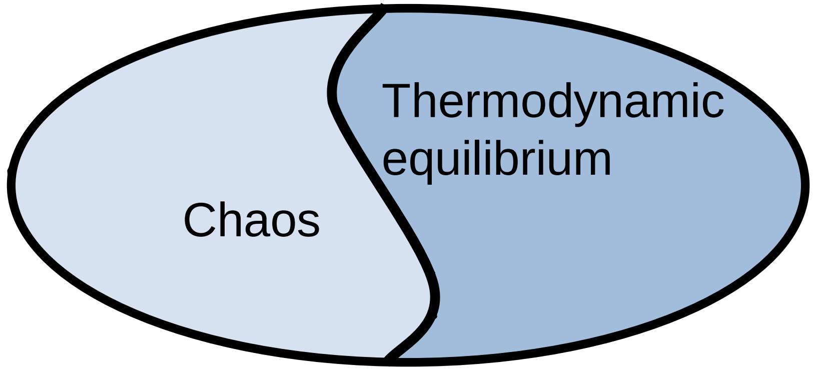
SDE



Probabilistic

ChT-SDE: Butterfly Effect

All natural DS's



-) Ground state has unstable or "unthermalized" variables (positive Lyapunov exponents), in which it is not a probability distribution

-) The "butterfly effect"

-) This is an ordered (or low-symmetry) phase – opposed to the semantics of word "chaos"

-) Statistics not applicable (example, replica trick)

-) The stationary total probability distribution is (among) the ground state(s). *Note: not the issue of its existence!*

-) Thermodynamics, statistics are applicable (Markov chains)

-) Forgets initial conditions/perturbations

Comparison with Supersymmetry in Quantum Theory:

-) Hadron Collider – the primary goal is to find supersymmetry in quantum theory of elementary particles

ChT-SDE: supersymmetry exists (at least) everywhere else from quantum theory

-) Supersymmetry (if exists) in quantum world must be spontaneously broken. Prioblems, however, with theory of susy breaking – susy's hard to break.

ChT-SDE: Chaos is the spontaneously broken susy.

In particular, all life forms are DS's with spontaneously broken susy

Importance Emphasized by Classics

Werner Heisenberg: " *When I meet God, I am going to ask him two questions: Why relativity and why turbulence? I really believe he will have an answer for the first.*"

Richard Feynman: "*turbulence – the most important unsolved problem of classical physics*"

Stephen Hawking: "*it is in complexity that I think the most important developments of the next millennium will be.*"

ChT-SDE: Previous Approach

$$\boxed{\partial_t x(t) = F(x(t)) + (2\Theta)^{1/2} e_a(x(t))\xi^a(t) \approx \mathfrak{F}(t)}$$

If $\overline{f}(t) = \int f(x)P(xt)d^D x$, what would be $\overline{f}(t+\Delta t)$ according to the SDE ?

This is how it derived: $x(t+\Delta t) = x(t) + \Delta x$,

Now $\overline{f}(t+\Delta t) = \left\langle \int f(x+\Delta x)P(xt)d^D x \right\rangle_{Ns}$

$= \left\langle \int \left(f(x) + \Delta x^i f_{,i}(x) + (1/2)\Delta x^i \Delta x^j f_{,ij}(x) + \ldots \right) P(xt)d^D x \right\rangle_{Ns} =:$

Further,

$\Delta x^i = \Delta t \mathfrak{F}(x^i + \alpha \Delta x^i) = \Delta t \mathfrak{F}^i(x) + \Delta t^2 \alpha \mathfrak{F}^i_{,j}(x)\mathfrak{F}^j(x) + \ldots$

(Ito, $\alpha = 0$, Stratonovich, $\alpha = 1/2$)

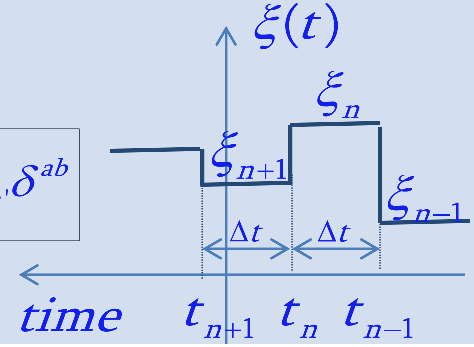
$:= \int f(x)\left(\hat{1} - \Delta t \hat{H}(\alpha) + \ldots\right) P(xt)d^D x$

Thus Fokker-Planck Equation is:

$\partial_t P = -\hat{H}(\alpha)P$

$$\overline{f}(t+\Delta t) = \left\langle \int f(M_{t+\Delta t,t}(x))P(xt)d^D x \right\rangle_{Ns}$$
$$= (M_{t+\Delta t,t}(x) \to x)$$
$$= \left\langle \int f(x) M^*_{t,t+\Delta t} P(xt)d^D x \right\rangle_{Ns}$$
$$= \int f(x) \left\langle M^*_{t,t+\Delta t} \right\rangle_{Ns} P(xt)d^D x$$

$$\left\langle \xi^a_n \xi^b_{n'} \right\rangle_{Ns} = \frac{1}{\Delta t}\delta_{nn'}\delta^{ab}$$



66

Chaotic Long-Range Order: Potential Origin

Long-Range Order = Gapless Excitation

Ubiquity of CLRO

Accidental picture

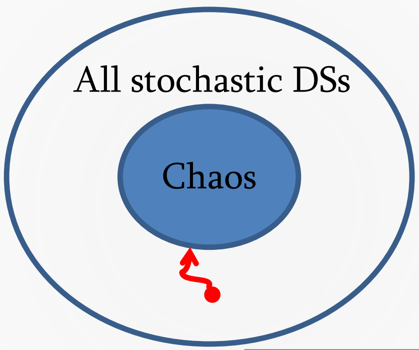
CLRO is a "critical" phenomenon - some excitation has zero gap because the DS is at a phase transition

Symmetry breaking picture:

CLRO is a symmetry breaking phenomenon - a gapless excitation is the Goldstone particle.

~~Contradiction~~

Requirement
All DSs must possess this (super)symmetry



Self-Organized Criticality (Bak&Tang&Wiesenfeld, 1987): proposition to believe that there is a <u>mysterious force</u> that self-tunes parameters of some SDEs to the phase transition into "chaos":

1) A mystery explain by another mystery. Technically, this is just a green light for using well-developed RG approach.
2) If this is true, where does the CLRO come from in chaotic DSs?
3) There was no definition of chaos for stochastic dynamics

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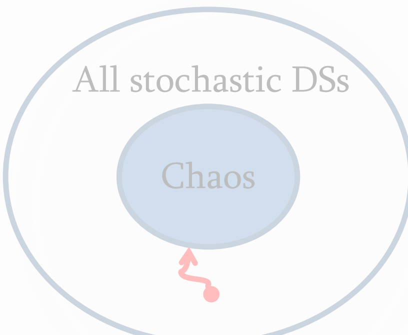

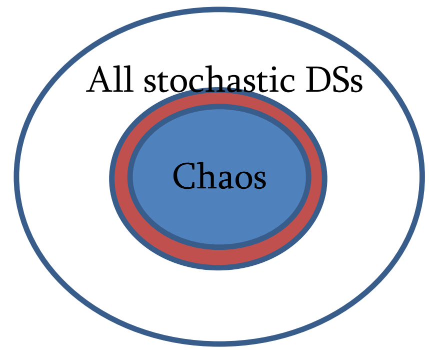

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Such a supersymmetry does exist !

-) Langevin SDEs: Parisi&Sourlas (1979)

-) Its topological meaning: Witten (1982) -> ChTs (1988).

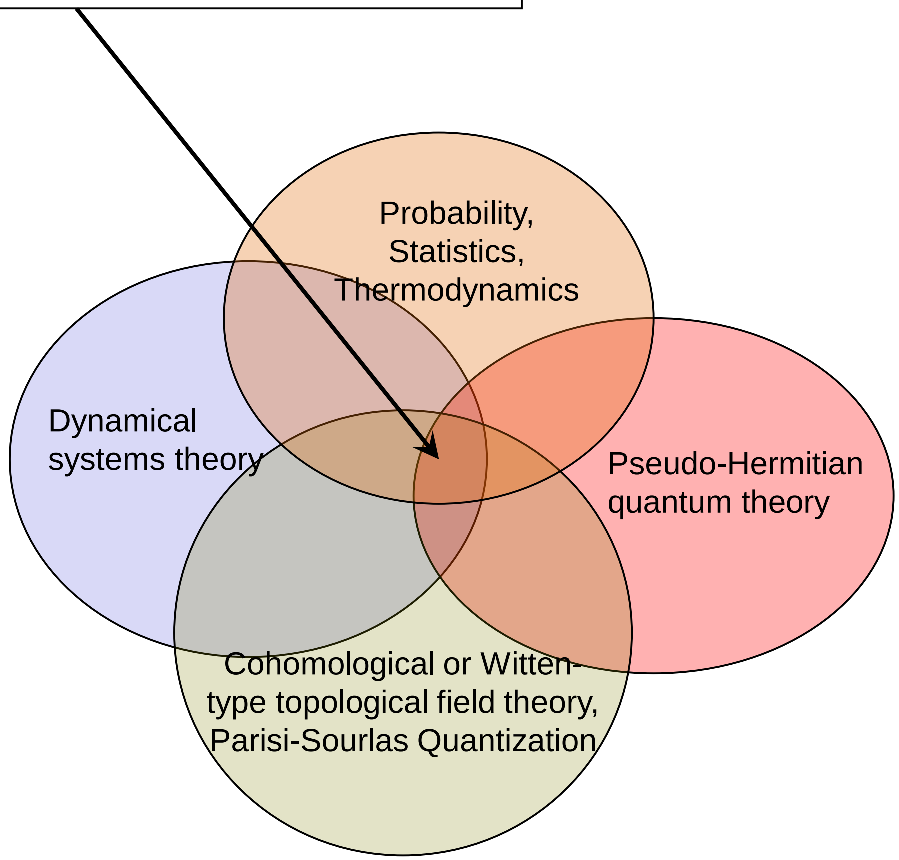
-) Pseudo-Hermitian Ev. Op. Mostafazadeh (2002)

-) Stochastic Evolution Op.: Ruelle (1987)

-) ChT-SDE for all SDEs (2013)

Geneology of ChT-SDE

ChT-SDE – approximation-free theory of SDEs



ChT-SDE – dynamical generalization of statistics, or

stochastic generalization of DS theory

Mathematical Foundation: Cohomological or Supersymmetric Field Theories