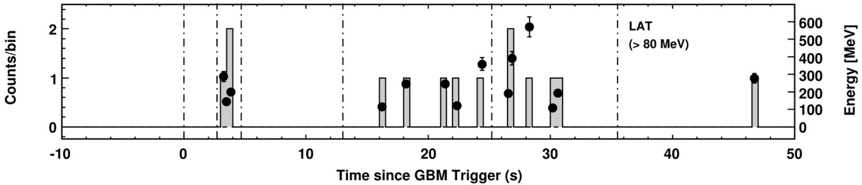


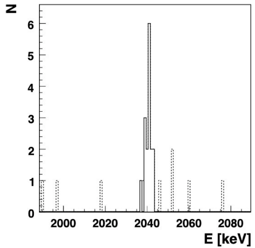

# On the On/Off Problem

summary of: [Knoetig2014]
title image: [Abdo et al.2009]

Max L. Ahnen

# Analysis logic

Discovery or not ?




Analyze energy spectrum and decide if there is evidence for a signal.
Counting experiment – Poisson statistics.

Figure: [Caldwell2012]

Inspiration:

[Caldwell and Kröninger2006]

Here: closed form special case.

# Analysis logic

Two step procedure:

- Find out significance of measurement
- Calculate signal credibility intervals if detection, or UL if not.

# The On/Off Measurement

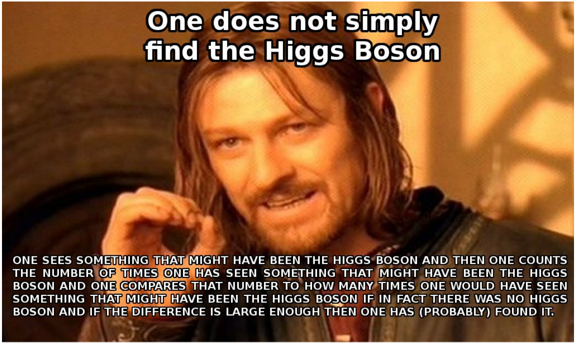



Figure: 9gag.com/gag/4792101

# The On/Off Measurement

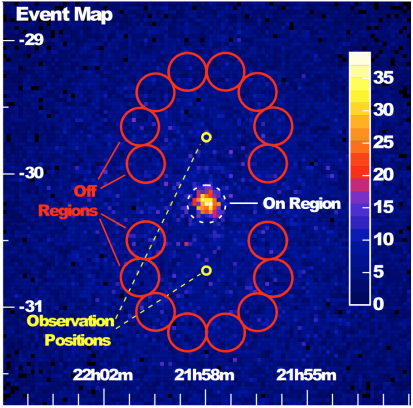



Figure: [Berge et al.2007]

# The On/Off Measurement

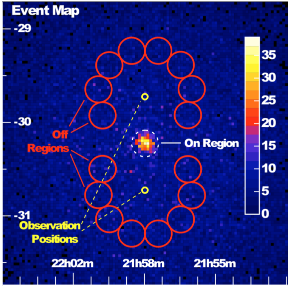



Figure: [Berge et al.2007]

Event numbers in bins follow Poisson distribution

$$P_P(N|\lambda) = \frac{e^{-\lambda}\lambda^N}{N!} \qquad (1)$$

# The On/Off Measurement

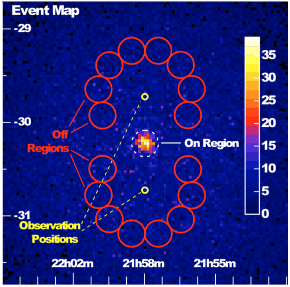



Figure: [Berge et al.2007]

Null hypothesis $H_0$ Likelihood (bg only)

$$P\left(N_{\mathrm{on}}, N_{\mathrm{off}} | \lambda_{\mathrm{bg}}, H_0\right) = P_{\mathrm{P}}\left(N_{\mathrm{on}} | \alpha\lambda_{\mathrm{bg}}\right) P_{\mathrm{P}}\left(N_{\mathrm{off}} | \lambda_{\mathrm{bg}}\right) \quad (2)$$

With a signal in the `On` region — $H_1$

$$P\left(N_{\mathrm{on}}, N_{\mathrm{off}} | \lambda_{\mathrm{s}}, \lambda_{\mathrm{bg}}, H_1\right) = P_{\mathrm{P}}\left(N_{\mathrm{on}} | \lambda_{\mathrm{s}} + \alpha\lambda_{\mathrm{bg}}\right) P_{\mathrm{P}}\left(N_{\mathrm{off}} | \lambda_{\mathrm{bg}}\right) \quad (3)$$

# Discovery or not?

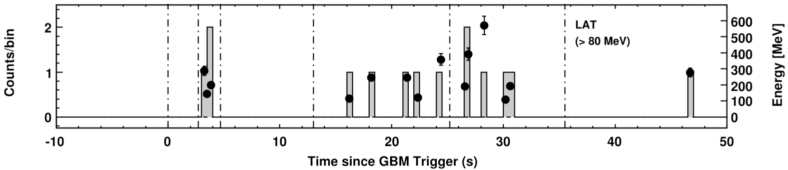



**Figure 1.** Light curves of GRB 080825C observed by the GBM (NaI & BGO) and LAT instruments; top two panels are background subtracted. The LAT light curve has been generated using events which passed the "S3" event selection above 80 MeV (which are also the events used for our spectral analysis). Black dots, along with their error bars (systematic uncertainty in the LAT energy measurement) represent the 1$\sigma$ energy range (right $y$-axis) for each LAT event. The vertical dash-dotted lines indicate the time bins used in our time-resolved spectral analysis.

Figure: [Abdo et al.2009]

# My problem

High-energy astrophysics: few counts. Available methods:

Bayesian

- Tail-area probability based (reject hypothesis without alternative)

  [Gillessen and Harney2005]

- Subjective Bayesian, introducing another parameter

  [Gregory2005]

Frequentist

- most not suitable for low count numbers

  [Li and Ma1983]

- trouble at the border of parameter space

  [Rolke et al.2005]

# My problem

None cover the full problem:

Significance of a measurement + Signal estimation

# Objective Bayesian Analysis

proceeds by

- Modeling initial uncertainty using ’non informative’ Priors, usually improper
- Using Bayes theorem to find proper posterior probability distributions, given data

# Objective Bayesian Analysis

Problems:

- There is basic agreement on objective Bayesian estimation
- Unfortunately no agreement on objective Bayesian hypothesis testing!

# Bayes Factors

Use Bayes rule for hypothesis testing

$$P(H_\mathrm{i}|N_\mathrm{on},N_\mathrm{off}) = \frac{P(N_\mathrm{on},N_\mathrm{off}|H_\mathrm{i})\,P_0(H_\mathrm{i})}{P(N_\mathrm{on},N_\mathrm{off})} \tag{4}$$

+ law of total probability => the Bayes factor $B_\mathrm{ij}$:

$$\begin{aligned}
\frac{P(H_\mathrm{i}|N_\mathrm{on},N_\mathrm{off})}{P(H_\mathrm{j}|N_\mathrm{on},N_\mathrm{off})} &= \frac{P(N_\mathrm{on},N_\mathrm{off}|H_\mathrm{i})\,P_0(H_\mathrm{i})}{P(N_\mathrm{on},N_\mathrm{off}|H_\mathrm{j})\,P_0(H_\mathrm{j})} \\
&= \frac{\int P\left(N_\mathrm{on},N_\mathrm{off}|\vec{\lambda}_i,H_i\right)P_0\left(\vec{\lambda}_i|H_i\right)d\vec{\lambda}_i}{\int P\left(N_\mathrm{on},N_\mathrm{off}|\vec{\lambda}_j,H_j\right)P_0\left(\vec{\lambda}_j|H_j\right)d\vec{\lambda}_j}\cdot\frac{P_0(H_\mathrm{i})}{P_0(H_\mathrm{j})} \\
&= B_\mathrm{ij}\cdot P_\mathrm{ij}
\end{aligned} \tag{5}$$

# Bayes Factors

Further assuming a complete set of $N$ exclusive rival hypothesis such that $P(H_i \wedge H_j) = 0$ for $i \neq j$ one can express the posterior model probability with Eqn. 5 conveniently as

$$P\left(H_{\mathrm{i}} | N_{\mathrm{on}}, N_{\mathrm{off}}\right) \quad = \quad \frac{B_{0\mathrm{i}}}{1 + \sum_{\mathrm{j}=1}^{N-1} B_{\mathrm{j}0} P_{\mathrm{j}0}} \tag{6}$$

..but in this case (On/Off problem, $\{H_0, H_1\}$) the difference between making decisions with Eqn. 5 and Eqn. 6 is small. One can also argue that there is always systematic errors and therefore no "complete set"

# Jeffreys's Rule

Harold Jeffreys revived the objective Bayesian view with his work, when people turned to Fisher tests, p-values, ... His suggestion was to use a prior that yielded the same answer, no matter what parametrization:

$$P_0\left(\vec{\lambda}_i|H_i\right) \propto \sqrt{\det\left[I\left(\vec{\lambda}_i|H_i\right)\right]}, \tag{7}$$

$$I_{kl}\left(\vec{\lambda}_i|H_i\right) = -E\left[\frac{\partial^2 \ln L\left(N_{\text{on}}, N_{\text{off}}|\vec{\lambda}_i, H_i\right)}{\partial\lambda_k \partial\lambda_l}\right], \tag{8}$$

where $I_{kl}$ denotes the Fisher information matrix, $L$ the likelihood function (either Eqn. 2 or 3), $E$ the expectation value with respect to the model with index $i$ and $\vec{\lambda}_i$

# Jeffreys's Rule — Benefits

- Almost always defined
- Transforms properly to give results indep. of parametrization
- Almost always gives proper posterior
- Often arises from more general treatments as limiting case

# Jeffreys's Rule — Weaknesses

- Violates the (controversial) likelihood principle that all evidence is embedded in the likelihood. -> but all objective Bayesian methods do
- Fisher information matrix must exist -> it does in this case [Knoetig2014]
- Can fail badly ( posterior does not converge to “true" result ) in higher-dimensional problems -> has to be checked!

# objective Bayes Factors?

Remember:

$$B_{\text{ij}} = \frac{\int P\left(N_{\text{on}}, N_{\text{off}}|\vec{\lambda}_i, H_i\right) P_0\left(\vec{\lambda}_i|H_i\right) d\vec{\lambda}_i}{\int P\left(N_{\text{on}}, N_{\text{off}}|\vec{\lambda}_j, H_j\right) P_0\left(\vec{\lambda}_j|H_j\right) d\vec{\lambda}_j} \tag{9}$$

but $P_0\left(\vec{\lambda}_i|H_i\right)$ from Jeffrey's rule is only defined up to constant $c_{\text{i}}$!

# objective Bayes Factors?

Suggestion: Variation of
[Spiegelhalter and Smith1981],[Ghosh and Samanta2002]

Imagine dataset with smallest physical sample size —
$N_{\text{on}} = N_{\text{off}} = 0$. Then

$$B_{01} = 1 + \epsilon; \|\epsilon\| = \text{rather small.} \tag{10}$$

(evidence that exists must be weak, because of 0 counts) ->
calibrate the undefined ratio of constants $\frac{c_i}{c_j}$!

# Objective Bayesian hypothesis testing — Summary assumptions

- Bayesian hypothesis testing in the On/Off problem with Bayes factors
- Jeffreys's rule prior for each model
- Calibrate the undefined constants by
  ”when you see nothing you do not learn (much)“

# Objective Bayesian hypothesis testing — Results

$$B_{01} = \frac{c_0}{c_1} \cdot \frac{\gamma}{\delta} \tag{11}$$

where

$$\gamma := (1 + 2N_{\text{off}})\, \alpha^{\frac{1}{2}+N_{\text{on}}+N_{\text{off}}} \cdot \Gamma\left(\frac{1}{2} + N_{\text{on}} + N_{\text{off}}\right) \tag{12}$$

$$\delta := 2\,(1+\alpha)^{N_{\text{on}}+N_{\text{off}}}\, \Gamma\left(1 + N_{\text{on}} + N_{\text{off}}\right) \cdot {}_2F_1\left(\frac{1}{2} + N_{\text{off}}, 1 + N_{\text{on}} + N_{\text{off}}; \frac{3}{2} + N_{\text{off}}; -\frac{1}{\alpha}\right) \tag{13}$$

$$\frac{c_0}{c_1} = \frac{2\arctan\left(\frac{1}{\sqrt{\alpha}}\right)}{\sqrt{\pi}} \tag{14}$$

[Knoetig2014]

# Objective Bayesian hypothesis testing — Results

Claim detection when Bayes factor $B_{01}$ is low. If the counted events lead to detection -> infer signal, assuming $H_1$!

# Objective Bayesian estimation

This is less controversial as the undefined constants cancel out.
Bayes rule gives:

$$P\left(\lambda_{\mathrm{s}}, \lambda_{\mathrm{bg}} | N_{\mathrm{on}}, N_{\mathrm{off}}, H_1\right) = \frac{P\left(N_{\mathrm{on}}, N_{\mathrm{off}} | \lambda_{\mathrm{s}}, \lambda_{\mathrm{bg}}, H_1\right) P_0\left(\lambda_{\mathrm{s}}, \lambda_{\mathrm{bg}} | H_1\right)}{\int_0^\infty \int_0^\infty P\left(N_{\mathrm{on}}, N_{\mathrm{off}} | \lambda_{\mathrm{s}}, \lambda_{\mathrm{bg}}, H_1\right) P_0\left(\lambda_{\mathrm{s}}, \lambda_{\mathrm{bg}} | H_1\right) d\lambda_{\mathrm{s}} d\lambda_{\mathrm{bg}}} \tag{15}$$

+ marginalization + Jeffreys's prior

# Objective Bayesian estimation — Results

$$P\left(\lambda_s \middle| N_{on}, N_{off}, H_1\right) = P_P\left(N_{on} + N_{off} \middle| \lambda_s\right) \cdot \frac{U\left[\frac{1}{2} + N_{off}, 1 + N_{off} + N_{on}, \left(1 + \frac{1}{\alpha}\right)\lambda_s\right]}{{}_2\tilde{F}_1\left(\frac{1}{2} + N_{off}, 1 + N_{off} + N_{on}; \frac{3}{2} + N_{off}; -\frac{1}{\alpha}\right)} \tag{16}$$

This can be used for quoting credibility intervals or, if the detection threshold was not reached, upper limits. [Caldwell and Kröninger2006],[Knoetig2014]

# First example: GRB080825C

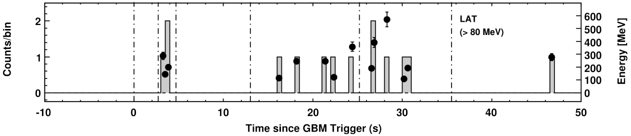



**Figure 1.** Light curves of GRB 080825C observed by the GBM (NaI & BGO) and LAT instruments; top two panels are background subtracted. The LAT light curve has been generated using events which passed the "S3" event selection above 80 MeV (which are also the events used for our spectral analysis). Black dots, along with their error bars (systematic uncertainty in the LAT energy measurement) represent the $1\sigma$ energy range (right $y$-axis) for each LAT event. The vertical dash-dotted lines indicate the time bins used in our time-resolved spectral analysis.

Figure: [Abdo et al.2009]

$N_{\text{on}} = 15, N_{\text{off}} = 19, \alpha = 33/525$ => $B_{01} = 9.66 \times 10^{-10}$ Detection!

# First example: GRB080825C

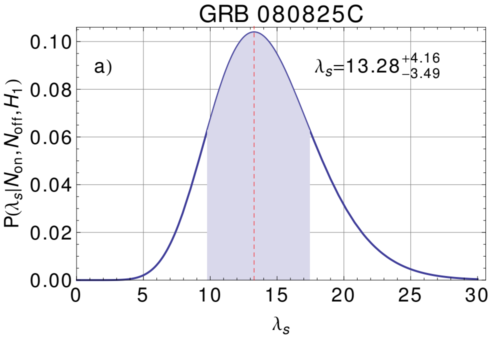



Figure: [Knoetig2014]

published value: $\lambda_s = 13.7$ [Abdo et al.2009]

# Second example: GRB080330

$N_{\text{on}} = 0, N_{\text{off}} = 15, \alpha = 0.123$ => $B_{01} = 2.29$ Upper Limit!

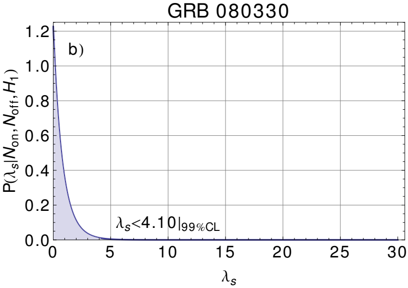



Figure: [Knoetig2014]

published value: $\lambda_s < 2.40$ [Acciari et al.2011]

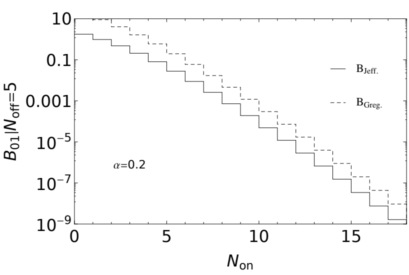
# Validation

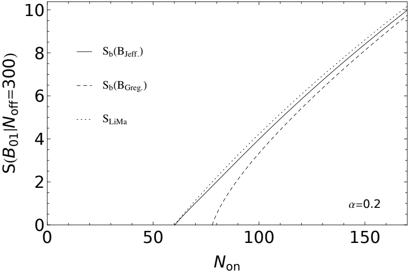
In order to compare to [Li and Ma1983] frequentist result use ”Bayesian“ z-value [Gillessen and Harney2005]

$$S_\mathrm{b} = \sqrt{2}\,\mathrm{erf}^{-1}\left[1 - B_{01}\right]. \quad (17)$$

$B_{01} = 5.7 \cdot 10^{-7}$ would correspond to ”5 sigma“

# Validation, hypothesis test




Few counts: compared to [Gregory2005], mostly within one count.
Many counts: close to frequentist result [Li and Ma1983]

# Validation, signal estimation

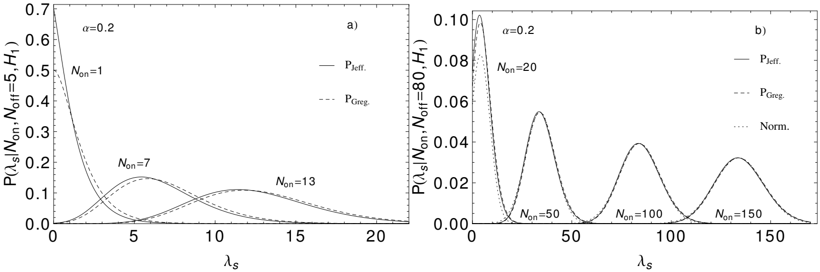



Figure: [Knoetig2014]

Similar to [Gregory2005], both converge to classical result in the many counts case.

# Conclusion

Claiming detections, setting credibility intervals, or setting upper limits can be unified over the whole On/Off problem parameter range in one consistent objective Bayesian method.

Sample implementation (Python, Mathematica):
https://polybox.ethz.ch/public.php?service=files&t=29958626f8bd78e4e8fbdbc7f7955c49

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