A Exercises 1

A.1 Magnitude of 4-velocity

Calculate the magnitude (vector length) of the 4-velocity u, defined by

$$u^\mu = \frac{dx^\mu}{d\tau}$$

- Use the definition of the 4-coordinates.
- Since this is a scaler, you can choose to work in any inertial frame - choose the easiest one.

A.2 4-acceleration

Show that the 4-acceleration is perpendicular to the 4-velocity w.r.t to the metric defined

$$a^\mu = \frac{d}{d\tau}\frac{p^\mu}{m_0}$$

A.3 Coordinate transformation to spherical coordinates

Calculate the metric tensor of Minkowski space-time in spherical coordinates, defined by

$$\begin{aligned} x &= r \sin\theta \sin\phi \\ y &= r \sin\theta \cos\phi \\ z &= r \cos\theta \end{aligned}$$

- Calculate the differentials dx, dy, dz by deriving the transformations with respect to the new variables (r, θ, ϕ). Note that $x = x(r, \theta, \phi)$ - you will use the chain rule.
- Plug your results into the line element ds^2 of the Minkowski space-time in Cartesian coordinates.
- By inspection of the result, find the metric tensor for Minkowski space-time in spherical coordinates.

A.4 Coordinate transformation to elliptical coordinates

Calculate the metric of Minkowski space-time for the transformation to elliptical coordinates, defined by

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin\theta \sin\phi \\ y &= \sqrt{r^2 + a^2} \sin\theta \cos\phi \\ z &= r \cos\theta \end{aligned}$$

For $a = 0$, this is the same as in exercise A.3. The path of calculation is the same.

A.5 Lorentz transformation

A Lorentz transformation is defined by $(c\,t, x, y, z) \longrightarrow (c\,t', x', y', z')$:

$$
\begin{aligned}
t' &= \gamma\left(t - \frac{v_x}{c^2}x\right) \\
x' &= \gamma(x - v_x t) \\
y' &= y \\
z' &= z
\end{aligned}
$$

Show that a space-time interval $|s'|$, i.e. the vector length of 4-vector s', is the same as for the original vector 's' - i.e. that vector lengths are invariant under Lorentz transformations.

A.6 Keplerian angular velocity

Calculate for a circular orbit around mass M the angular velocity ω, which is a function of radius.

A.7 Effective potential & Keplerian orbits

Re-educate yourself on Keplerian orbits in a Newtonian potential, central mass M.

- Start from $E = E_{\mathrm{kin}} + E_{\mathrm{pot}}$
- Express the velocity v in a radial and tangential component
- Introduce the angular momentum $l = r^2\dot{\phi}$
- Read off the form of the effective potential $V_{\mathrm{eff}}(r)$
- What are the turning points of the motion, as a function of E and l? $(dr/dt = 0)$
- In the energy equation, transform from $\dot{r}$ to $\frac{dr}{d\phi}$
- Transform once again to $u = 1/r$
- Take the derivative $\frac{d}{d\phi}$
- The second order differential equation for $u(\phi)$ obtained is "easy". Can you guess the solution?
- Verify that $u(\phi) = 1 + e\cos\phi$ solves the equation
- What are the turning points, as a function of l and e ?