J Exercises 10

J.1 Kepler's law in the Kerr metric

Assume an equatorial, circular orbit in the Kerr metric: $dr = 0$, $\theta = \pi/2$ and $d\theta = 0$.

- To what simplifies the metric in this case? $ds^2 = ...$?
- Derive this equation with respect to r.
- The resulting equation can be solved for $\omega = \frac{d\phi}{dt}$, which yields thus Kepler's third law for the Kerr metric
- Verify that for $a = 0$ the result agrees with what we found for the Schwarzschild metric.

J.2 Precession of orbits in the Kerr metric

A circular, non-equatorial orbit precesses around the spin axis with an angular velocity of

$$\Omega_{\text{prec}} = \frac{2GJ}{c^2 r^3} \tag{794}$$

Evaluate this for two cases:

- A satellite orbiting Earth in 200 km height.
- The star S2 around Sgr A*, assuming a maximally spinning black hole ($\chi = 1$). The orbit of S2 has $a =$ 125 mas, and the system is located in a distance of 8.3 kpc.

J.3 Derivation of ergosphere

For the derivation of the ergosphere, one needs to show that these two expressions are equivalent:

$$\begin{aligned} &(r^2 - r\,r_S + a^2)(r^2 + a^2\cos^2\theta)^2 + a^2\,r^2\,r_S^2\sin^2\theta \\ &(r^2 + a^2\cos^2\theta - rrs)((r^2 + a^2)^2 - a^2(r^2 - rrs + a^2)\sin^2\theta) \end{aligned} \tag{795}$$

Show that!

J.4 Eddington luminosity

The Eddington luminosity is the luminosity at which the radiation pressure equals the gravity on the infalling gas.

$$F_{\text{grav}} = \vec{\nabla}\Phi = \frac{\sigma_{\text{T}}}{c\,m_{\text{P}}} F_{\text{rad}} \tag{796}$$

Assume Newtonian gravity with the Poisson equation $\vec{\nabla}^2\Phi = 4\pi G\rho$. Evaluate

$$L_{\text{edd}} = \int_S F_{\text{rad}}\,dS \tag{797}$$

(using Gauss' law to convert the surface integral into a volume integral).