1 Historic primer

Imagine, you want to leave a celestial body with radius R. What speed does one need? The total energy E of the rocket needs be positive:

$$
\begin{aligned}
E &= \frac{1}{2}mv^2 + V(R) \\
0 &= \frac{1}{2}mv^2 - \frac{GmM}{R} \\
v &= \sqrt{\frac{2GM}{R}} \qquad (1)
\end{aligned}
$$

For Earth, this is 11.2 km/s, for the Sun 618 km/s. Apparently, the bigger body has a larger escape speed. Now imagine, the sphere gets larger, at constant density (unrealistic!) - at what radius would the escape speed reach the speed of light?

$$
\begin{aligned}
c &= \sqrt{\frac{2G \cdot 4/3\pi r^3 \rho_\odot}{r}} \\
&= r\sqrt{\frac{8\pi G}{3}\rho_\odot} \\
r &= 485 R_\odot = 2.5\,\mathrm{AU} \qquad (2)
\end{aligned}
$$

From such a star, light would not be able to escape, and it would need to be black. It was John Michell, a British natural philosopher, who wrote down this thought experiment first in 1783. In his words:
If there should really exist in nature any bodies, whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us; or if there should exist any other bodies of a somewhat smaller size, which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from light; yet, if any other luminous bodies should happen to revolve about them we might still perhaps from the motions of these revolving bodies infer the existence of the central ones with some degree of probability, [...].
Independent and without knowing from from Michell, Simon Laplace wrote in 1796 in his "Exposition du Système du Monde":
The gravitation attraction of a star with a diameter 250 times that of the Sun and comparable in density to the earth would be so great no light could escape from its surface. The largest bodies in the universe may thus be invisible by reason of their magnitude.
He also provided a mathematical proof, i.e. a calculation similar to the above.
It took than more than 100 years until Albert Einstein in 1916 published the "General theory of relativity", and in the same year Karl Schwarzschild published (to the surprise of Einstein) an exact solution to Einstein's equations. The solution worked beautifully for the solar system, but it also predicted that compact objects would be dark stars. But they considered it more a curiosity of the theory than a reality of nature.
It was only in the 1960's that the topic came into focus of the scientific community.

- 1939, work from Richard Tolman, Robert Oppenheimer and George Volkoff showed that an upper limit for the mass of a neutron star exists for it to be stable. For heavier, compact objects no stabilizing force against gravity is known.
- In 1963, the New Zealander Roy Kerr presented a solution that corresponds to a rotating black hole.

- In 1965, Roger Penrose showed that black holes actually can form (and are not an artefact of the symmetry assumed in the calculations). The key concept was that of ”trapped surfaces”, which was honored with the 2020 Nobel prize in physics.
- 1967 that the American physicist John Wheeler coined the term ’black hole’, replacing the term ’completely collapsed objects’.
- Donald Lynden-Bell and Martin Rees proposed in 1971 that in every galaxy an active ot dormant, massive black hole resides - and also in our Milky Way.
- 1972 Tom Bolton was able to convincingly identify the first stellar-mass black hole in the Milky Way: Cygnus X-1
- In 2002 the team around Reinhard Genzel determined the mass of Sgr A* from the orbit of a star around it, excluding essentially all other possibilities than that it is a massive black hole with 4 million solar masses.
- 2015 the LIGO gravitational wave experiment discovered its first event, a merger of two black holes of 29 and 36 solar masses
- In 2019, the event horizon telescope collaboration published its first resolved image of a black hole, in the center of the galaxy M87, with a mass of 6.5 billion solar masses.

The scope of this lecture is to understand classical black holes, and get to know the key observations of these objects. One groups black holes typically by mass

- Particle physics scale: Black holes with masses reachable via particle accelerators, or from cosmic ray interactions. Due to their Hawing radiation these should be very bright emitters and short-lived. We don’t have any experimental evidence for their existence
- Primordial black holes: Black holes could have formed directly during the big bang. While masses below 4×10^{11} kg should have evaporated since the big bang, masses larger than that would still be around. In particular, planetary masses (10^{24} kg) are being discussed as possible dark matter candidates. No direct evidence for these black holes has been found.
- Stellar mass black holes have been found in many stellar systems - historically mostly in binary systems, where unseen companions were sometimes too heavy to be a neutron star; sometimes also with accretion disks visible in the X-ray regime. Nowadays such black holes are also seen in gravitational waves, when two such objects merge.
- Intermediate mass black holes: Beyond a few 100 up to $10^5\,M_\odot$ there is some marginal evidence for such black holes, mostly in globular clusters. These objects are attractive to explain the even heavier counterparts in merger trees.
- (super-) massive black holes: Almost all galaxies host in their centers a massive black hole, the mass of which scales with galaxy properties. The most prominent example is Sgr A* in our own Milky Way.

’Black Holes’ is a booming field of research (figure 1), and it has diversified into many subbranches. On the theoretical side, black holes might be the entry into the world of quantum gravity, which is beyond the scope of this lecture. On the observational side, black holes are building blocks of the universe, with important roles in galaxy formation and growth regulation. The Galactic Center black hole is used for tests of general relativity, and gravitational wave detections question formation channels of stellar-mass black holes.

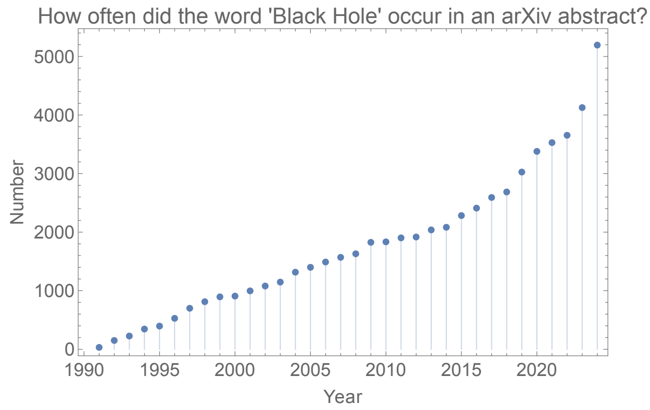


Figure 1: The number of occurrences of the word 'Black Hole' in abstracts submitted to arXiv as a function of year.

2 Tensor algebra - the maths of general relativity

Here is a collection of useful definitions and relations, introducing also the canonic notation for general relativity.

2.1 Euclidean, Cartesian coordinates

Coordinates are

$$\vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = \sum_{i=1}^{3} x^i\vec{e}_i = x^i\vec{e}_i \tag{3}$$

Note that the numbers to top right of the x are not "to the power of", but coordinate indices. Some care and understanding is needed, when one reads a symbol like x^2. For the unit vectors we have:

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \tag{4}$$

The Kronecker-δ is here the 3D unity matrix. The Euclidean dot product is

$$\begin{aligned} \vec{x}\cdot\vec{y} &= x_1y_1 + x_2y_2 + x_3y_3 \\ &= \sum_{i=1}^{3} x_iy_i = x_i\, y^i \\ &= (x_1, x_2, x_3)\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= \vec{x}\cdot g\cdot\vec{y} = g_{ij}\, x^i\, y^i \end{aligned} \tag{5}$$

Note the Einstein summation convention: Indices which appear twice, once upper and once lower, are automatically summed over. One can always change the name of such an index pair, as it is "dummy" (like an integration variable in an integral). This notation also gives a convenient way of checking validity of equations: Both sides need to have the same indices in upper and lower positions, after the summations are executed. Further, one has for tensors

$$A^i{}_i = g^{ij}A_{ji} = \mathrm{tr}(A^i{}_j) \tag{6}$$

The use of latin letters for indices indicates 3D, space vectors.

2.2 4D space-time coordinates

The coordinates are

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} \text{ or for spherical coordinates} : x^\mu = \begin{pmatrix} ct \\ r \\ \theta \\ \phi \end{pmatrix} \tag{7}$$

And with the unit vector e_μ the vector x is:

$$x = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 = \sum_{\mu=0}^{3} x^\mu e_\mu = x^\mu e_\mu \tag{8}$$

For 4-vectors, we use greek indices.

2.2.1 Minkowski space

The dot product $x \cdot y$ is defined via

$$x \cdot y = \eta_{\mu\nu} x^\mu y^\mu \tag{9}$$

with the Minkowski metric:

$$\eta_{\mu\nu}^{\text{Cartesian}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \eta_{\mu\nu}^{\text{Spherical}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{10}$$

For a line element one has

$$\begin{aligned} ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \end{aligned} \tag{11}$$

Note that there are different conventions found in the literature: The signs of the metric might be opposite, the c^2 could be part of the metric or in the definition of the 0-components of the coordinates (as here), or even an imaginary i is used sometimes to express the opposite sign of the time component compared to the spatial components.

2.2.2 Curved space-time coordinates

The components of a vector are as usual

$$\begin{aligned} x &= x^\mu e_\mu \\ dx &= dx^\mu e_\mu \end{aligned} \tag{12}$$

with the novelty that the base vectors e_μ are not constants, but can be functions of the coordinates. The dot product gets generalized by going from $\eta_{\mu\nu}$ to $g_{\mu\nu}$, which can also be of a more complicated functional form. The dot product for curved manifolds is defined via

$$\begin{aligned} A(x) \cdot B(x) &= g_{\mu\nu}(x) A^\mu(x) B^\nu(x) = A_\nu(x) B^\nu(x) \\ A \cdot B &= g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu \end{aligned} \tag{13}$$

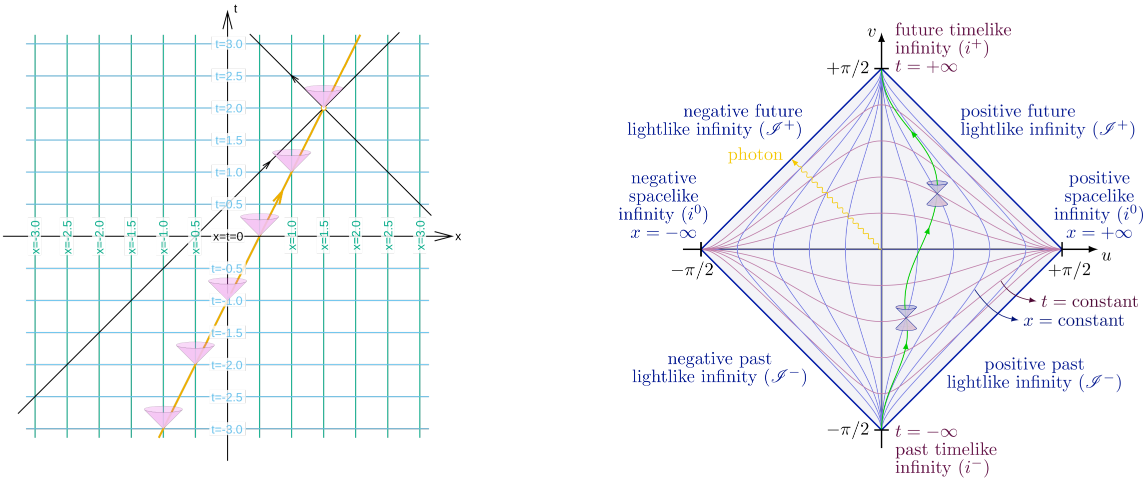


Figure 2: Two representation of a (1-1) Minkowski space-time. Left: Flat coordinates. The worldline of an observer at constant velocity is shown, together with the lightcones, giving the regions of space-time, which causally can connect to the respective event. Light rays travel diagonally, time-like trajectories are more vertical than the light rays. Right: Again Minkowski space-time, plotted in a Penrose diagram with coordinates (u, v), in which are defined by $r + ct = \tan(u + v)$, $r - ct = \tan(u - v)$. This form of the diagram is useful to describe black holes later. Source: German Wikipedia and TikZ.

Note that the two vectors need to be evaluated at the same space-time point. So one can always calculate a vector length, but (in general) not the cross product of two space vectors X, Y The metric tensor $g_{\mu\nu}$ is

$$\begin{aligned} g_{\mu\nu} &= e_\mu e_\nu = e_\nu e_\mu = g_{\nu\mu} \\ \delta^\rho_\nu &= e^{\,\rho}\, e_{\,\nu} = g^{\rho\mu} e_{\,\mu}\, e_{\,\nu} = g^{\rho\mu} g_{\mu\nu} = g^\rho_\nu \\ g_{\mu\nu} &= (g^{\mu\nu})^{-1} \end{aligned} \quad (14)$$

The metric tensor is thus symmetric. The last line follows from the second line, noting that the Kronecker-δ here is the 4D unity matrix. With that one gets the line element

$$ds^2 = (dx^\mu e_\mu)(dx^\nu e_\nu) = e_\mu e_\nu dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \quad (15)$$

which is thus a generalization of the usual Euclidean Pythagorean theorem for infinitesimal paths in curved space-time. The coordinate transformation (changing $x \longrightarrow x'$) of a vector A ($\longrightarrow A'$) is

$$\begin{aligned} A'^\alpha &= \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \\ A'_\alpha &= \frac{\partial x^\beta}{\partial x'^\alpha} A_\beta \end{aligned} \quad (16)$$

For the example of velocity V:

$$V'^\alpha = \frac{\partial x'^\alpha}{\partial \tau} = \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial x^\beta}{\partial \tau} = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta \quad (17)$$

2.2.3 Note on derivatives

Derivatives are written as:

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \vec{\nabla} \end{pmatrix} \tag{18}$$

The ∂_μ is a very useful notation. In many books, one also finds the notation $X_{\mu,\nu}$ for $\partial_\nu X_\mu$. Here, we don't use it. The 4-gradient is

$$\begin{aligned} \nabla &= (\partial_0, \partial_1, \partial_2, \partial_3) \\ \nabla_{\text{Cartesian}} &= (\frac{1}{c}\partial_t, \partial_x, \partial_y, \partial_z) \\ \nabla_{\text{Spherical}} &= (\frac{1}{c}\partial_t, \partial_r, \partial_\theta, \partial_\phi) \end{aligned} \tag{19}$$

And the square of the 4-gradient (the Laplace operator) is:

$$\nabla^2 = g^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu \tag{20}$$

It requires thus the inverse of the metric tensor.

2.2.4 Equivalence principle

The equivalence principle states that one can gets the same experimental results in any reference frame, i.e. one can also choose a free-falling one, in which locally no gravity is felt, and hence for free falling reference systems one has $g_{\mu\nu} = \eta_{\mu\nu}$.

2.2.5 Example: Magnitude of 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{d}{d\tau}\vec{x} \end{pmatrix} = \begin{pmatrix} c\gamma \\ \frac{d}{dt}\vec{x}\frac{dt}{d\tau} \end{pmatrix} = \begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix} \tag{21}$$

Going to a local inertial frame, one can use the Minkowski metric.

$$\begin{aligned} |u|^2 &= u_\nu u^\nu = \eta_{\mu\nu}u^\mu u^\nu = (c\gamma, \vec{v}\gamma).\begin{pmatrix} -1 & 0 \\ \vec{0} & \mathbb{1} \end{pmatrix}.\begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix} = -c^2\gamma^2 + \vec{v}^2\gamma^2 \\ &= (v^2 - c^2)\frac{1}{1-\frac{v^2}{c^2}} = (v^2 - c^2)\frac{c^2}{c^2 - v^2} = -c^2 \end{aligned} \tag{22}$$

Since scalars are Lorentz-invariant, this result holds in any reference frame.

3 Relativistic dynamics

Curves are often parametrized by the "proper time" τ, i.e. the time passing for a particle moving along the space-time curve. τ in general differs from the coordinate time t. This defines the Lorentz factor:

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{23}$$

Here, $v = |\vec{v}|$. For a free-falling observer in his local inertial frame $x' = (ct', x'^0, x'^1, x'^3) = (c\tau, \mathrm{const}, \mathrm{const}, \mathrm{const})$, i.e he is at constant coordinates, and time passes at the "proper time". For him, the line elements reads thus

$$ds^2 = -c^2 d\tau^2 \tag{24}$$

which must hold in any reference frame, as it is a scalar relation. This relation holds for any real-world particle, and one calls this a "time-like" trajectory. For light, one has $ds^2 = 0$, leading to "null geodesics".

3.1 Energy and momentum

For particles with rest mass m_0 the 4-momentum p is:

$$p = m_0 u = m_0 \begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} \tag{25}$$

Note that E and $\vec{p}$ are different than in Newtonian mechanics. In Minkowski space, its norm is

$$|p|^2 = p_\mu p^\mu = \eta_{\nu\mu} p^\nu p^\mu = -m_0^2 \gamma^2 c^2 + m_0^2 \gamma^2 v^2 = -m_0^2 \gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = -m_0^2 \gamma^2 c^2 \frac{1}{\gamma^2} = -m_0^2 c^2 \tag{26}$$

The 4-acceleration is

$$a^\mu = \frac{d}{d\tau} \frac{p^\mu}{m_0} \tag{27}$$

4-velocity and 4-acceleration are orthogonal to each other (when m_0 is constant):

$$u.a = \eta_{\nu\mu} \frac{p^\nu}{m_0} a^\mu = \eta_{\nu\mu} \frac{p^\nu}{m_0} \frac{d}{d\tau} \frac{p^\mu}{m_0} = \frac{1}{2m_0^2} \frac{d}{d\tau} p^2 = \frac{1}{2m_0^2} \frac{d}{d\tau} (-m_0^2 c^2) = 0 \tag{28}$$

The energy can be written also in this form:

$$E = -\eta_{\nu\mu} p^\nu u^\mu = -m_0\, \eta_{\nu\mu} u^\nu u^\mu = m_0\, c^2 \tag{29}$$

But this equation is also valid, if one measures the energy of a particle moving with u in a system moving with v:

$$E = -\eta_{\nu\mu} p^\nu v^\mu = -m_0\, \eta_{\nu\mu} u^\nu v^\mu \tag{30}$$

This definition will be carried over to general relativity:

$$E = -g_{\nu\mu} p^\nu u^\mu \tag{31}$$

The kinetic energy K is:

$$K = E - m_0 c^2 = (\gamma - 1) m_0 c^2 \tag{32}$$

Using the series expansion for γ

$$\gamma \approx 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} \tag{33}$$

one sees that the leading order of K is

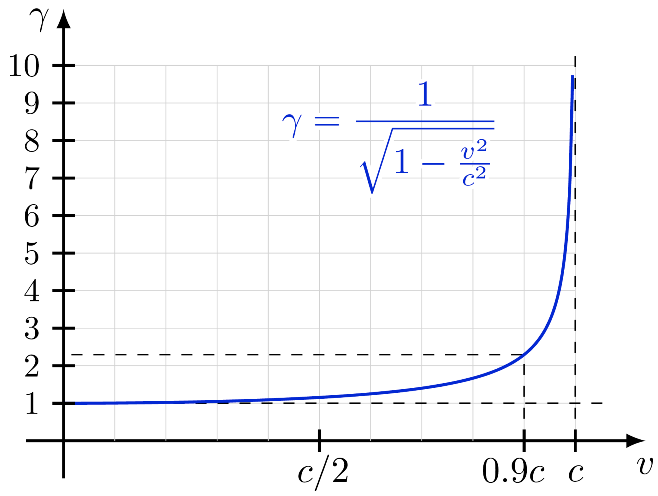
$$K = (\gamma - 1) m_0 c^2 \approx \frac{1}{2} m_0 v^2 \tag{34}$$

From the definitions of E and $\vec{p}$ follows:

$$\begin{aligned}
\vec{v}E/c &= \vec{p}c \\
\frac{v^2}{c^2}E^2 &= (|\vec{p}|c)^2 \\
E^2(1-\frac{v^2}{c^2}) &= E^2-(|\vec{p}|c)^2 \\
E^2/\gamma^2 &= E^2-(|\vec{p}|c)^2 \\
m_0c^2 &= E^2-(|\vec{p}|c)^2
\end{aligned} \tag{35}$$

This is the relativistic energy-momentum relation. Newton's second law takes the form

$$\vec{F}=\frac{d\vec{p}}{dt} \tag{36}$$



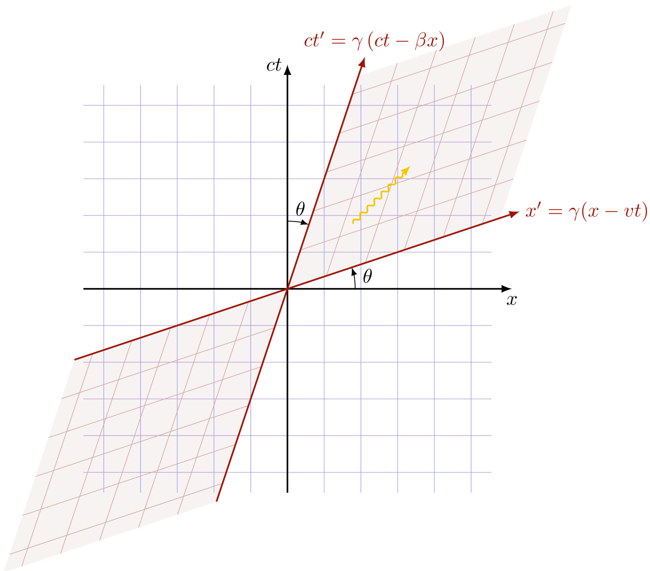


Figure 3: Left: The Lorentz-factor γ as a function of velocity. Right: A Lorentz-transformation of the Minkowski-space time. Source: TikZ

3.2 Lorentz transformations

For an observer moving relative to another one along the x-axis with velocity v_x, the coordinates are

$$\begin{aligned}
t' &= \gamma\left(t-\frac{v_x}{c^2}x\right) \\
x' &= \gamma(x-v_xt) \\
y' &= y \\
z' &= z
\end{aligned} \tag{37}$$

It is easy to calculate

$$v'_x = \frac{dx'}{dt'} = \frac{d}{dt'}\gamma(x - v_x t) = \frac{d}{dt'}\gamma\left(x - v_x\left(\frac{t'}{\gamma} + \frac{v_x}{c^2}x\right)\right) = -v_x \tag{38}$$

In matrix form, one can write:

$$\begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \frac{v}{c}\gamma & 0 & 0 \\ \frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} \tag{39}$$

A space-time interval is invariant under a Lorentz transformation:

$$\begin{aligned} s'^2 &= -(c\,t')^2 + x'^2 + y'^2 + z'^2 \\ &= -c^2\gamma^2\left(t - \frac{v_x}{c^2}x\right)^2 + \gamma^2(x - v_x t)^2 + y^2 + z^2 \\ &= \gamma^2(-c^2t^2 + 2c^2\frac{v_x}{c^2}x - \frac{v_x^2}{c^2}x^2) + \gamma^2(x^2 - 2x\,v_x t + v_x^2t^2) + y^2 + z^2 \\ &= \gamma^2c^2t^2(-1 + \frac{v_x^2}{c^2}) + \gamma^2x^2(1 - \frac{v_x^2}{c^2}) + y^2 + z^2 \\ &= \gamma^2c^2t^2\frac{-1}{\gamma^2} + \gamma^2x^2\frac{1}{\gamma^2} + y^2 + z^2 \\ &= -(c\,t)^2 + x^2 + y^2 + z^2 = s^2 \end{aligned} \tag{40}$$

The transformation law for velocities follows: The velocity v'_x measured in the primed coordinate system that moves with velocity u is given by the original v_x and u by:

$$v'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - udt)}{\gamma(dt - \frac{u}{c^2dx})} = \frac{\frac{dx}{t} - u}{1 - \frac{u}{x^2}\frac{dx}{t}} = \frac{v_x - u}{1 - \frac{u\,v_x}{c^2}} \tag{41}$$

The same transformation can be applied to the energy-momentum vector $(E/c, \vec{p})$ or the wave vector $(\omega/c, \vec{k})$. Let's take E as an example:

$$\begin{aligned}
\frac{E'}{c} &= \frac{1}{c}\frac{m_0c^2}{\sqrt{1-v_x'^2/c^2}} \\
&= \frac{1}{c}\frac{m_0c^2}{\sqrt{1-\left(\frac{v_x-u}{1-\frac{u\,v_x}{c^2}}\right)^2/c^2}} \\
&= \frac{1}{c}\frac{m_0c^2\left(1-\frac{u\,v_x}{c^2}\right)}{\sqrt{\left(1-\frac{u\,v_x}{c^2}\right)^2-\frac{(v_x-u)^2}{c^2}}} \\
&= \frac{1}{c}\frac{m_0c^2\left(1-\frac{u\,v_x}{c^2}\right)}{\sqrt{1-2\frac{u\,v_x}{c^2}+\frac{u^2\,v_x^2}{c^4}-\frac{v_x^2}{c^2}+2\frac{uv_x}{c^2}-\frac{u^2}{c^2}}} \\
&= \frac{1}{c}\frac{m_0c^2\left(1-\frac{u\,v_x}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v_x^2}{c^2}\right)}} \\
&= \frac{1}{c}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(\gamma m_0c^2-u\,\gamma m_0v_x\right) \\
&= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(\frac{E}{c}-\frac{u}{c}\,p_x\right) \qquad (42)
\end{aligned}$$

So, this is the same as if we would have transformed $(E/c, \vec{p})$ by the same Lorentz boost by u as we did for x.

4 Curvature

4.1 Describing curvature

"If I move a vector into the direction of another vector, how do its components change" - a non-trivial question in curved space-time. For a base vector e_μ moving infinitesimal into direction x^ν, the four components for the e_λ are given by the Christoffel symbols (which are not tensors):

$$\Gamma^\lambda_{\ \mu\nu} = \frac{\partial e_\mu}{\partial x^\nu}\,e^{\,\lambda} \qquad (43)$$

This allows defining a covariant derivative: "The vector field A not only changes as a function of coordinates, but due to the curvature, there is also a change due to the coordinates changing."

$$\begin{aligned}
\nabla_\mu A^\lambda &= \partial_\mu A^\lambda + \Gamma^\lambda_{\ \mu\nu}A^\nu \\
\nabla_\mu A_\lambda &= \partial_\mu A_\lambda - \Gamma^\nu_{\ \mu\lambda}A_\nu \qquad (44)
\end{aligned}$$

We will not use the notation $A^\lambda_{\ ;\mu} = \nabla_\mu A^\lambda$. For a tensor, one has:

$$\nabla_\mu A^{\alpha\beta} = \partial_\mu A^{\alpha\beta} + \Gamma^\alpha_{\ \lambda\mu}A^{\lambda\beta} + \Gamma^\beta_{\ \mu\lambda}A^{\alpha\lambda} \qquad (45)$$

For GR space-time, a property called metric compatibility holds: The covariant derivative of the metric tensor is always zero:

$$\nabla_\lambda g^{\alpha\mu} = 0 \tag{46}$$

For a torsion-free space-time, we expect that derivatives commute: $\nabla_\mu \nabla_\nu = \nabla_\nu \nabla_\mu$. For a scalar f in flat coordinates the covariant derivative is the partial derivative, and clearly

$$\partial_\mu \partial_\nu f \quad = \quad \partial_\nu \partial_\mu f \tag{47}$$

If the symmetry holds in one coordinate system, it holds in all, hence

$$\begin{aligned} \nabla_\mu \partial_\nu f &= \nabla_\nu \partial_\mu f \\ \partial_\mu \partial_\nu f - \Gamma^\alpha{}_{\mu\nu} \partial_\alpha f &= \partial_\nu \partial_\mu f - \Gamma^\alpha{}_{\nu\mu} \partial_\alpha f \end{aligned} \tag{48}$$

and we see that the metric tensor needs to be symmetric in the second and third index.

$$\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu} \tag{49}$$

The Christoffel symbols can be expressed with the metric tensor. This requires some algebra. From the definition we have (by multiplying with e_λ and using $e_\lambda e^\lambda = 1$:

$$\begin{aligned} e_\lambda \Gamma^\lambda{}_{\alpha\nu} &= e_\lambda \Gamma^\lambda{}_{\nu\alpha} = e_\lambda \frac{\partial e_\nu}{\partial x^\alpha} e^\lambda = \frac{\partial e_\nu}{\partial x^\alpha} = \partial_\alpha e_\nu \\ e_\mu e_\lambda \Gamma^\lambda{}_{\alpha\nu} &= e_\mu \partial_\alpha e_\nu \end{aligned} \tag{50}$$

Consider

$$\partial_\alpha (e_\mu e_\nu) \quad = \quad e_\nu \partial_\alpha e_\mu + e_\mu \partial_\alpha e_\nu \quad \longrightarrow \quad e_\mu \partial_\alpha e_\nu = \partial_\alpha (e_\mu e_\nu) - e_\nu \partial_\alpha e_\mu \tag{51}$$

With that we get:

$$e_\mu e_\lambda \Gamma^\lambda{}_{\alpha\nu} \quad = \quad \partial_\alpha (e_\mu e_\nu) - e_\nu e_\lambda \Gamma^\lambda{}_{\alpha\mu} \tag{52}$$

Writing this equation two more times, but with indices relabelled:

$$e_\alpha e_\lambda \Gamma^\lambda{}_{\nu\mu} \quad = \quad \partial_\nu (e_\alpha e_\mu) - e_\mu e_\lambda \Gamma^\lambda{}_{\nu\alpha} \tag{53}$$

$$e_\alpha e_\lambda \Gamma^\lambda{}_{\mu\nu} \quad = \quad \partial_\mu (e_\alpha e_\nu) - e_\nu e_\lambda \Gamma^\lambda{}_{\mu\alpha} \tag{54}$$

Taking 52 + 53 - 54 yields, using $\Gamma^\lambda{}_{\nu\mu} = \Gamma^\lambda{}_{\mu\nu}$ and $\Gamma^\lambda{}_{\alpha\mu} = \Gamma^\lambda{}_{\mu\alpha}$:

$$\begin{aligned} e_\mu e_\lambda \Gamma^\lambda{}_{\alpha\nu} &= \partial_\alpha (e_\mu e_\nu) + \partial_\nu (e_\alpha e_\mu) - \partial_\mu (e_\alpha e_\nu) - e_\mu e_\lambda \Gamma^\lambda{}_{\nu\alpha} \\ 2 e_\mu e_\lambda \Gamma^\lambda{}_{\alpha\nu} &= \partial_\alpha (e_\mu e_\nu) + \partial_\nu (e_\alpha e_\mu) - \partial_\mu (e_\alpha e_\nu) \\ (e^\mu e^\rho)(e_\mu e_\lambda) \Gamma^\lambda{}_{\alpha\nu} &= \frac{1}{2} (e^\mu e^\rho)(\partial_\alpha (e_\mu e_\nu) + \partial_\nu (e_\alpha e_\mu) - \partial_\mu (e_\alpha e_\nu)) \\ \delta^\rho_\lambda \Gamma^\lambda{}_{\alpha\nu} = \Gamma^\rho{}_{\alpha\nu} &= \frac{1}{2} g^{\mu\rho} (\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu}) \\ \Gamma^\lambda{}_{\mu\nu} &= \frac{1}{2} g^{\lambda\alpha} (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \end{aligned} \tag{55}$$

For the following the Riemann curvature tensor is important:

$$R^{\lambda}{}_{\alpha\nu\mu} = \partial_\nu \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\mu \Gamma^{\lambda}{}_{\alpha\nu} + \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\alpha\mu} - \Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\alpha\nu} \tag{56}$$

Here it is defined with the "Riemann" sign convention. It has the following symmetries:

$$\begin{aligned} R_{\lambda\alpha\nu\mu} &= -R_{\alpha\lambda\mu\nu} \\ R^{\lambda}{}_{\alpha\nu\mu} &= -R^{\lambda}{}_{\alpha\mu\nu} \\ R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \end{aligned} \tag{57}$$

One obtains the so-called Ricci tensor as contraction from the Riemann tensor, or expressed in terms of Christoffel symbols:

$$\begin{aligned} R_{\alpha\mu} &= R^{\lambda}{}_{\alpha\lambda\mu} \\ &= \partial_\lambda \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\mu \Gamma^{\lambda}{}_{\alpha\lambda} + \Gamma^{\lambda}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\alpha\mu} - \Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\alpha\lambda} \end{aligned} \tag{58}$$

The Ricci tensor is symmetric

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} = g^{\beta\alpha} R_{\alpha\nu\beta\mu} = R^{\beta}_{\nu\beta\mu} = R_{\nu\mu} \tag{59}$$

The Ricci scalar is a contraction of the Ricci tensor:

$$R = R^{\alpha}_{\alpha} \tag{60}$$

4.2 Deriving the Bianchi identities

Working for the moment in a special coordinate system - the result will be a scalar, and hence independent of the choice of coordinate system. Going to a local inertial frame. Then, locally the metric is Minkowski, and the Christoffel symbols vanish. But: not the derivatives of the Christoffel symbols! "One can transform away locally the gravitational force, but not the tidal forces". Also, in a local inertial frame, the the covariant derivative becomes a partial derivative. The Riemann curvature in a local inertial frame is:

$$R^{\lambda}{}_{\alpha\nu\mu} = \partial_\nu \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\mu \Gamma^{\lambda}{}_{\alpha\nu} \tag{61}$$

Taking the covariant derivative:

$$\nabla_\sigma R^{\lambda}{}_{\alpha\nu\mu} = \partial_\sigma \partial_\nu \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\sigma \partial_\mu \Gamma^{\lambda}{}_{\alpha\nu} \tag{62}$$

Writing this three times, cycling indices:

$$\begin{aligned} \nabla_\sigma R^{\lambda}{}_{\alpha\nu\mu} &= \partial_\sigma \partial_\nu \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\sigma \partial_\mu \Gamma^{\lambda}{}_{\alpha\nu} \\ \nabla_\nu R^{\lambda}{}_{\alpha\mu\sigma} &= \partial_\nu \partial_\mu \Gamma^{\lambda}{}_{\alpha\sigma} - \partial_\nu \partial_\sigma \Gamma^{\lambda}{}_{\alpha\mu} \\ \nabla_\mu R^{\lambda}{}_{\alpha\sigma\nu} &= \partial_\mu \partial_\sigma \Gamma^{\lambda}{}_{\alpha\nu} - \partial_\mu \partial_\nu \Gamma^{\lambda}{}_{\alpha\sigma} \end{aligned} \tag{63}$$

Adding the three equations, and noting that partial derivatives commute, the right hand sides yields simply 0, which is known as the Bianchi identity

$$\nabla_\sigma R^{\lambda}{}_{\alpha\nu\mu} + \nabla_\nu R^{\lambda}{}_{\alpha\mu\sigma} + \nabla_\mu R^{\lambda}{}_{\alpha\sigma\nu} = 0 \tag{64}$$

This result is valid is any reference frame, as it is a tensor relation.

5 Geodesic equation

5.1 Standard approach

A geodesic is the equivalent of a straight line in Euclidean or Minkowski space. A body with no acceleration will move along such a line, and it is defined by velocity being constant:

$$0 = \frac{d\vec{u}}{d\tau} = \frac{d(u^\mu \vec{e}_\mu)}{d\tau} = \vec{e}_\mu \frac{du^\mu}{d\tau} + u^\mu \frac{d\vec{e}_\mu}{d\tau} \tag{65}$$

From this the geodesic equation can be derived:

$$\begin{aligned}
\vec{e}_\mu \frac{du^\mu}{d\tau} &= -u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
\vec{e}_\nu \vec{e}_\mu \frac{du^\mu}{d\tau} &= -\vec{e}_\nu u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
g_{\mu\nu} \frac{du^\mu}{d\tau} &= -\vec{e}_\nu u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
\frac{du_\nu}{d\tau} &= -\vec{e}_\nu u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
g^{\nu\lambda} \frac{du_\nu}{d\tau} &= -g^{\nu\lambda}\, \vec{e}_\nu\, u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
\frac{du^\lambda}{d\tau} &= -\vec{e}^{\,\lambda}\, u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
\frac{du^\lambda}{d\tau} &= -\vec{e}^{\,\lambda}\, u^\mu \frac{d\vec{e}_\mu}{dx^\nu} \frac{dx^\nu}{d\tau} \\
\frac{du^\lambda}{d\tau} &= -\frac{\partial \vec{e}_\mu}{\partial x^\nu}\, \vec{e}^{\,\lambda}\, u^\mu u^\nu \\
\frac{du^\lambda}{d\tau} &= -\Gamma^\lambda{}_{\mu\nu} u^\mu u^\nu
\end{aligned} \tag{66}$$

Hence, we have the two equivalent forms:

$$\begin{aligned}
\frac{du^\lambda}{d\tau} + \Gamma^\lambda{}_{\mu\nu} u^\mu u^\nu &= 0 \\
\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} &= 0
\end{aligned} \tag{67}$$

It is written here for proper time τ, but actually holds for any parameter.

5.2 Lagrangian formulation

There is an elegant way to express the geodesic equation using the Lagrangian $\mathcal{L}$.

$$\mathcal{L}(x, u) = -mc\sqrt{-g_{\mu\nu}(x) u^\mu u^\nu} \tag{68}$$

Note that formally $\mathcal{L} = -m\,c^2$. Geodesics are extremal in the sense, that they are the shortest paths between two points. The $\sqrt{X}$ is extremal when X is extremal, and hence one can also vary

$$\mathcal{L}'(x, u) = g_{\mu\nu}(x) u^\mu u^\nu \tag{69}$$

As usual, the coordinates x and velocities u are treated as independent variables. The Euler-Lagrange equation is then

$$\frac{\partial \mathcal{L}'}{\partial x^\lambda} = \frac{d}{d\tau}\frac{\partial \mathcal{L}'}{\partial u^\lambda} \tag{70}$$

This is identical to the geodesic equation. The left side is

$$\frac{\partial \mathcal{L}'}{\partial x^\lambda} = \frac{1}{2}(\partial_\lambda g_{\mu\nu})u^\mu u^\nu \tag{71}$$

The right side is

$$\begin{aligned}
\frac{d}{d\tau}\frac{\partial \mathcal{L}'}{\partial u^\lambda} &= \frac{d}{d\tau}\left(\frac{1}{2}g_{\mu\nu}\delta^\mu_\lambda u^\nu + \frac{1}{2}g_{\mu\nu}u^\mu\delta^\nu_\lambda\right) \\
&= \frac{d}{d\tau}\left(g_{\mu\lambda}u^\mu\right) \\
&= \partial_\nu g_{\mu\lambda}u^\nu u^\mu + g_{\mu\lambda}\ddot{x}^\mu \\
&= \frac{1}{2}\partial_\nu g_{\mu\lambda}u^\nu u^\mu + \frac{1}{2}\partial_\mu g_{\nu\lambda}u^\mu u^\nu + g_{\mu\lambda}\ddot{x}^\mu
\end{aligned} \tag{72}$$

In the last step, the previous term was split and summation indices have been renamed. The dot-derivative indicates deriving with respect to proper time. Putting things together

$$\begin{aligned}
\frac{1}{2}\partial_\lambda g_{\mu\nu}u^\mu u^\nu &= \frac{1}{2}\partial_\nu g_{\mu\lambda}u^\nu u^\mu + \frac{1}{2}\partial_\mu g_{\nu\lambda}u^\mu u^\nu + g_{\mu\lambda}\ddot{x}^\mu \\
0 &= g_{\mu\lambda}\ddot{x}^\mu + \frac{1}{2}\partial_\nu g_{\mu\lambda}u^\nu u^\mu + \frac{1}{2}\partial_\mu g_{\nu\lambda}u^\mu u^\nu - \frac{1}{2}\partial_\lambda g_{\mu\nu}u^\mu u^\nu \\
0 &= \ddot{x}_\lambda + \Gamma_{\lambda\mu\nu}u^\mu u^\nu \\
0 &= g^{\lambda\rho}\ddot{x}_\lambda + g^{\lambda\rho}\Gamma_{\lambda\mu\nu}u^\mu u^\nu \\
0 &= \ddot{x}^\rho + \Gamma^\rho{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu
\end{aligned} \tag{73}$$

An advantage of this formulation is that often it avoids using Christoffel symbols.

5.2.1 The choice of Lagrangian

Since we optimized $\mathcal{L}'$, why don't we call that Lagrangian? The reason is, that the action

$$S_\tau = \int d\tau \mathcal{L} \tag{74}$$

shall be invariant under transformations from one affine parameter τ to another one σ:

$$\begin{aligned}
S_\sigma &= -mc\int d\sigma\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}} \\
&= -mc\int \frac{d\sigma}{d\tau}d\tau\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{d\tau^2}{d\sigma^2}} \\
&= -mc\int d\tau\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = S_\tau
\end{aligned} \tag{75}$$

This does not hold for $\mathcal{L}'$, but the choice of $\mathcal{L}$ in equation 68 is invariant under re-parametrization.

6 Energy-momentum tensor

Recap of continuity equation for charge (expressing conservation of the scalar charge):

$$\frac{d\rho}{dt} + \vec{\nabla}\vec{j} = 0 \tag{76}$$

Using the 4-current $j^\mu = (c\rho, \vec{j})$ the continuity equation gets

$$\partial_\mu j^\mu = 0 \tag{77}$$

Conserving a scalar quantity is thus an equation with a gradient of a 4-vector. The four-momentum is

$$p^\mu = \begin{pmatrix} \frac{E}{c} \\ \gamma m_0 \vec{v} \end{pmatrix} \tag{78}$$

Conserving energy and momentum, i.e. the four-momentum, being a vector, is thus an equation with a gradient of some $4x4$ tensor:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{79}$$

Symbolically, the components (in a local inertial frame) are:

$$T = \partial_V \begin{pmatrix} E & p & p & p \\ p & P & S & S \\ p & S & P & P \\ p & S & S & P \end{pmatrix} \tag{80}$$

Note the T_{00} component: It is the energy density, which is the relativistic version of mass density (as in Newtonian mechanics), and all energy components need to be taken into account. So, for the Newtonian limit, $T_{00} = \rho c^2$, which also holds for pressure-less dust. p refers to momentum, P to pressure, and S to shear. The form of T for various cases can easily be looked up, for example for a perfect fluid.

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu + P g^{\mu\nu} \tag{81}$$

In a local inertial frame, this is a diagonal matrix with diagonal $(\rho c^2, P, P, P)$.

7 Derivation of the field equations

7.1 Finding the right tensors

- Independence of coordinate system choice: Tensor equations
- Spacetime curvature should relate to matter
- In a weak-field, slow motion limit, one should recover Newton's equations
- locally, energy and momentum should be conserved

We thus look for a tensor equation. The simplest form in which curvature can occur is a two-index curvature tensor. Matter, in the simplest form, can then be represented by the energy-momentum tensor $T^{\mu\nu}$. So we postulate some equation of type $G^{\mu\nu} \propto T^{\mu\nu}$, with an unknown tensor G. As energy and momentum shall be conserved, i.e. $\nabla_\mu T^{\mu\nu} = 0$, one also has to demand that $\nabla_\mu G^{\mu\nu} = 0$. And we can construct one, starting from the Bianchi identity:

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \tag{82}$$

Multiplying with $g^{\alpha\mu}$ and swapping indices in the second term:

$$g^{\alpha\mu}\nabla_\lambda R_{\alpha\beta\mu\nu} - g^{\alpha\mu}\nabla_\nu R_{\alpha\beta\mu\lambda} + g^{\alpha\mu}\nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \tag{83}$$

Due to the metric compatibility, one can move the $g^{\alpha\mu}$ into the derivatives, where it will raise the α index to μ (first two terms) and:

$$\nabla_\lambda R^\mu{}_{\beta\mu\nu} - \nabla_\nu R^\mu{}_{\beta\mu\lambda} + \nabla^\alpha R_{\alpha\beta\nu\lambda} = 0 \tag{84}$$

In the first and second term, the Ricci tensor appeared, and swapping indices in the last term:

$$\nabla_\lambda R_{\beta\nu} - \nabla_\nu R_{\beta\lambda} - \nabla^\alpha R_{\beta\alpha\nu\lambda} = 0 \tag{85}$$

Multiplying with $g^{\beta\nu}$ and again using the metric compatibility, raising indices, and substituting Ricci tensor and scalar when they occur, one gets

$$\begin{aligned} \nabla_\lambda g^{\beta\nu} R_{\beta\nu} - \nabla_\nu g^{\beta\nu} R_{\beta\lambda} - \nabla^\alpha g^{\beta\nu} R_{\beta\alpha\nu\lambda} &= 0 \\ \nabla_\lambda R^\nu_\nu - \nabla_\nu R^\nu_\lambda - \nabla^\alpha R^\nu_{\alpha\nu\lambda} &= 0 \\ \nabla_\lambda R - \nabla_\nu R^\nu_\lambda - \nabla^\alpha R_{\alpha\lambda} &= 0 \end{aligned} \tag{86}$$

Using $\nabla^\alpha = g^{\alpha\rho}\nabla_\rho$ and using again the metric compatibility:

$$\begin{aligned} \nabla_\lambda R - \nabla_\nu R^\nu_\lambda - \nabla_\rho g^{\alpha\rho} R_{\alpha\lambda} &= 0 \\ \nabla_\lambda R - \nabla_\nu R^\nu_\lambda - \nabla_\rho R^\rho_\lambda &= 0 \end{aligned} \tag{87}$$

Here, ν and ρ are just summation indices, so the second and third term are the same.

$$\nabla_\lambda R - 2\nabla_\nu R^\nu_\lambda = 0 \tag{88}$$

Multiplying with $g^{\mu\lambda}$, and again using the metric compatibility and raising indices:

$$\begin{aligned} \nabla_\lambda g^{\mu\lambda} R - 2\nabla_\nu g^{\mu\lambda} R^\nu_\lambda &= 0 \\ \nabla_\lambda g^{\mu\lambda} R - 2\nabla_\nu R^{\mu\nu} &= 0 \\ \nabla_\nu g^{\mu\nu} R - 2\nabla_\nu R^{\mu\nu} &= 0 \\ 2\nabla_\nu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) &= 0 \end{aligned} \tag{89}$$

Define $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$, the last equation states $\nabla_\mu G^{\mu\nu} = 0$. Hence, G is a divergence free, two-index curvature tensor - so it is a viable candidate for the field equation. It is the simple-most such choice. Therefore, we can guess here, that the field equations are

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu} \tag{90}$$

7.2 Weak fields and slow motions

Note that there is no way to prove the field equations, as they are the theory Einstein **postulated**. But one can connect them to classical mechanics for the limit of a weak field (i.e. slow motions), which can serve at least as a motivation, as it shows that in this limit, we recover what we know.

7.2.1 Connection to Newtonian potential

First, it is useful to see how the Einstein tensor $G^{\mu\nu}$ connects to the Newton potential in the case of a weak field. For a weak field, the metric should be almost Minkowski, such that one can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{91}$$

with a small $h \ll 1$. For this weak-field metric, the Christoffel symbols are

$$\begin{aligned}
\Gamma^{\lambda}{}_{\mu\nu} &= \frac{1}{2}(\eta^{\lambda\alpha} + h^{\lambda\alpha})\left[\partial_\mu(\eta_{\alpha\nu} + h_{\alpha\nu}) + \partial_\nu(\eta_{\alpha\mu} + h_{\alpha\mu}) - \partial_\alpha(\eta_{\mu\nu} + \epsilon h_{\mu\nu})\right] \\
&= \frac{1}{2}\eta^{\lambda\alpha}(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}) \\
&= \frac{1}{2}(\partial_\mu h^{\lambda}{}_{\nu} + \partial_\nu h^{\lambda}{}_{\mu} - \partial^{\lambda} h_{\mu\nu})
\end{aligned} \tag{92}$$

where higher order terms of h are dropped, and the derivatives of the constant Minkowski metric vanish. The Newtonian equations only have spatial derivatives, and thus only $i = 1, 2, 3$ of the 00-component matter:

$$\Gamma^{i}{}_{00} = \frac{1}{2}(\partial_0 h^{i}{}_{0} + \partial_0 h^{i}{}_{0} - \partial^{i} h_{00}) \tag{93}$$

The time derivatives ∂_0 are 0, as our metric is constant. It remains the spatial one, and we get

$$\Gamma^{i}{}_{00} \approx -\frac{1}{2}\partial^{i} h_{00} \tag{94}$$

Note that since h is small, so is $\Gamma^{i}{}_{00}$. Next, one evaluates the (spatial part of the) geodesic equation. We are dealing with slow motions, $v \ll c$. In that limit, $\tau \to t$, and hence

$$\frac{d^2x^i}{dt^2} + \Gamma^{i}{}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 \tag{95}$$

The velocity terms $\frac{dx^i}{dt}$ are much smaller than $\frac{dx^0}{dt} = c$ and thus

$$\begin{aligned}
\frac{d^2x^i}{dt^2} + \Gamma^{i}{}_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} &= 0 \\
\frac{d^2x^i}{dt^2} + c^2\Gamma^{i}{}_{00} &= 0
\end{aligned} \tag{96}$$

As expected, only the $\Gamma^{i}{}_{00}$ Christoffel symbol appears. Using the calculation from above

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2\partial^{i} h_{00} = \frac{1}{2}c^2\vec{\nabla}^{i} h_{00} \tag{97}$$

In Newton's theory we have

$$\frac{d^2x^i}{dt^2} = -\vec{\nabla}^i\Phi \tag{98}$$

Therefore, we can identify

$$\begin{aligned} h_{00} &= -\frac{2\Phi}{c^2} \\ g_{00} &= -1 - \frac{2\Phi}{c^2} \end{aligned} \tag{99}$$

7.2.2 Proportionality constant

In Newtonian gravity, the only source of gravity is mass or energy density, which corresponds to the T_{00}-component of the energy-momentum tensor, and all other components being 0:

$$T_{00} = \rho c^2 \tag{100}$$

Hence, we only need to look at the 00-component for determining the proportionality constant κ.

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = \kappa\rho c^2 \tag{101}$$

Using the Ricci tensor in the form expressed as Christoffel symbols, one sees that in the weak field limit, the third and fourth term can be neglected, as they are squares of Γ, with $\Gamma \sim h \ll 1$. Hence one has

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\ \mu\nu} - \partial_\nu\Gamma^\alpha_{\ \mu\alpha} \tag{102}$$

The 00-component is

$$R_{00} = \partial_\alpha\Gamma^\alpha_{\ 00} - \partial_0\Gamma^\alpha_{\ 0\alpha} \tag{103}$$

As our metric tensor is not time variable, ∂_0 yields 0. Thus

$$R_{00} = \partial_i\Gamma^i_{\ 00} - \partial_0\Gamma^0_{\ 00} \tag{104}$$

where the last term again is 0. Then

$$R_{00} = \partial_i\left(-\frac{1}{2}\partial^i h_{00}\right) = -\frac{1}{2}\partial_i\partial^i\left(-\frac{2\Phi}{c^2}\right) = \frac{1}{c^2}\vec{\nabla}^2\Phi \tag{105}$$

Given that only the T_{00} component is present in the Newtonian limit, the spatial components G_{ij} are 0:

$$R_{ij} - \frac{1}{2}g_{ij}R = 0 \longrightarrow R_{ij} = \frac{1}{2}g_{ij}R \tag{106}$$

Splitting up the contraction sum of the Ricci scalar:

$$R = R^\nu_\nu = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij} = g^{00}R_{00} + \frac{1}{2}g^{ij}g_{ij}R = g^{00}R_{00} + \frac{3}{2}R \tag{107}$$

where the identity $g^{ij}g_{ij} = 3$ has been used. This can be solved for R:

$$R = -2g^{00}R_{00} \tag{108}$$

With $g^{00} = 1/g_{00}$ the Ricci tensor is

$$R = -2\left(\frac{1}{-1-\frac{2\Phi}{c^2}}\right)\frac{1}{c^2}\vec{\nabla}^2\Phi \approx 2\left(1+\frac{2\Phi}{c^2}\right)\frac{1}{c^2}\vec{\nabla}^2\Phi \approx \frac{2}{c^2}\vec{\nabla}^2\Phi + O(\Phi^2) \tag{109}$$

We thus have R and R_{00}, and can evaluate the 00-component of the field equation:

$$\begin{aligned}
R_{00} - \frac{1}{2}g_{00}R &= \kappa T_{00} \\
\frac{1}{c^2}\vec{\nabla}^2\Phi - \frac{1}{2}\left(-1-\frac{2\Phi}{c^2}\right)\frac{2}{c^2}\vec{\nabla}^2\Phi &= \kappa\rho c^2 \\
\frac{1}{c^2}\vec{\nabla}^2\Phi + \frac{1}{2}\left(1+\frac{2\Phi}{c^2}\right)\frac{2}{c^2}\vec{\nabla}^2\Phi &= \kappa\rho c^2 \\
\frac{1}{c^2}\vec{\nabla}^2\Phi + \frac{1}{2}\frac{2}{c^2}\vec{\nabla}^2\Phi + O(\Phi^2) &= \kappa\rho c^2 \\
\frac{2}{c^2}\vec{\nabla}^2\Phi &= \kappa\rho c^2 \\
\vec{\nabla}^2\Phi &= \frac{1}{2}\kappa\rho c^4
\end{aligned} \tag{110}$$

Newtonian gravity is given by the Poisson equation

$$\vec{\nabla}^2\Phi = 4\pi G\rho \tag{111}$$

and hence $\kappa = \frac{8\pi G}{c^4}$. Finally, the Einstein field equations are

$$\begin{aligned}
G_{\mu\nu} &= \frac{8\pi G}{c^4}T_{\mu\nu} \\
G^{\mu\nu} &= \frac{8\pi G}{c^4}T^{\mu\nu}
\end{aligned} \tag{112}$$

7.3 Summary

What was the path to get to this equation?

- We started with some assumptions: We look for a tensor relation between curvature and matter that conserves energy and momentum, and that recovers Newton's equations.
- The simple-most choice is $G_{\mu\nu} = \kappa T_{\mu\nu}$.
- So we need a tensor for G, for which the covariant derivative vanishes - as $\nabla T = 0$.
- Using the Bianchi identity, we were able to construct one, consisting of the Ricci tensor and Ricci scalar. This required some assumptions on the metric: That it is torsion-free and that the metric compatibility holds.
- Using the geodesic equation of motion for a weak field / slow motion, and neglecting higher order terms, we saw that the 00-component of the metric relates to the Newtonian potential.
- With that we looked at the 00-equation of the proposed tensor relation, and were able to derive κ such that the Newtonian law is recovered in first order.
- That completes the proposal for the field equations. On whether this was a correct proposal, only experiment can judge.