# 1 Historic primer

Imagine, you want to leave a celestial body with radius $R$. What speed does one need? The total energy $E$ of the rocket needs be positive:

$$\begin{aligned} E &= \frac{1}{2}mv^2 + V(R) \\ 0 &= \frac{1}{2}mv^2 - \frac{GmM}{R} \\ v &= \sqrt{\frac{2GM}{R}} \end{aligned} \tag{1}$$

For Earth, this is 11.2 km/s, for the Sun 618 km/s. Apparently, the bigger body has a larger escape speed. Now imagine, the sphere gets larger, at constant density (unrealistic!) - at what radius would the escape speed reach the speed of light?

$$\begin{aligned} c &= \sqrt{\frac{2G \cdot 4/3\pi r^3 \rho_\odot}{r}} \\ &= r\sqrt{\frac{8\pi G}{3}\rho_\odot} \\ r &= 485 R_\odot = 2.5\,\mathrm{AU} \end{aligned} \tag{2}$$

From such a star, light would not be able to escape, and it would need to be black. It was John Michell, a British natural philosopher, who wrote down this thought experiment first in 1783. In his words:
*If there should really exist in nature any bodies, whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us; or if there should exist any other bodies of a somewhat smaller size, which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from light; yet, if any other luminous bodies should happen to revolve about them we might still perhaps from the motions of these revolving bodies infer the existence of the central ones with some degree of probability, [...].*
Independent and without knowing from from Michell, Simon Laplace wrote in 1796 in his "Exposition du Système du Monde":
*The gravitation attraction of a star with a diameter 250 times that of the Sun and comparable in density to the earth would be so great no light could escape from its surface. The largest bodies in the universe may thus be invisible by reason of their magnitude.*
He also provided a mathematical proof, i.e. a calculation similar to the above.
It took than more than 100 years until Albert Einstein in 1916 published the "General theory of relativity", and in the same year Karl Schwarzschild published (to the surprise of Einstein) an exact solution to Einstein's equations. The solution worked beautifully for the solar system, but it also predicted that compact objects would be dark stars. But they considered it more a curiosity of the theory than a reality of nature.
It was only in the 1960's that the topic came into focus of the scientific community.

- 1939, work from Richard Tolman, Robert Oppenheimer and George Volkoff showed that an upper limit for the mass of a neutron star exists for it to be stable. For heavier, compact objects no stabilizing force against gravity is known.
- In 1963, the New Zealander Roy Kerr presented a solution that corresponds to a rotating black hole.

- In 1965, Roger Penrose showed that black holes actually can form (and are not an artefact of the symmetry assumed in the calculations). The key concept was that of "trapped surfaces", which was honored with the 2020 Nobel prize in physics.
- 1967 that the American physicist John Wheeler coined the term 'black hole', replacing the term 'completely collapsed objects'.
- Donald Lynden-Bell and Martin Rees proposed in 1971 that in every galaxy an active ot dormant, massive black hole resides - and also in our Milky Way.
- 1972 Tom Bolton was able to convincingly identify the first stellar-mass black hole in the Milky Way: Cygnus X-1
- In 2002 the team around Reinhard Genzel determined the mass of Sgr A* from the orbit of a star around it, excluding essentially all other possibilities than that it is a massive black hole with 4 million solar masses.
- 2015 the LIGO gravitational wave experiment discovered its first event, a merger of two black holes of 29 and 36 solar masses
- In 2019, the event horizon telescope collaboration published its first resolved image of a black hole, in the center of the galaxy M87, with a mass of 6.5 billion solar masses.

The scope of this lecture is to understand classical black holes, and get to know the key observations of these objects. One groups black holes typically by mass

- Particle physics scale: Black holes with masses reachable via particle accelerators, or from cosmic ray interactions. Due to their Hawing radiation these should be very bright emitters and short-lived. We don't have any experimental evidence for their existence
- Primordial black holes: Black holes could have formed directly during the big bang. While masses below $4 \times 10^{11}$ kg should have evaporated since the big bang, masses larger than that would still be around. In particular, planetary masses ($10^{24}$ kg) are being discussed as possible dark matter candidates. No direct evidence for these black holes has been found.
- Stellar mass black holes have been found in many stellar systems - historically mostly in binary systems, where unseen companions were sometimes too heavy to be a neutron star; sometimes also with accretion disks visible in the X-ray regime. Nowadays such black holes are also seen in gravitational waves, when two such objects merge.
- Intermediate mass black holes: Beyond a few 100 up to $10^5\, M_\odot$ there is some marginal evidence for such black holes, mostly in globular clusters. These objects are attractive to explain the even heavier counterparts in merger trees.
- (super-) massive black holes: Almost all galaxies host in their centers a massive black hole, the mass of which scales with galaxy properties. The most prominent example is Sgr A* in our own Milky Way.

'Black Holes' is a booming field of research (figure 1), and it has diversified into many subbranches. On the theoretical side, black holes might be the entry into the world of quantum gravity, which is beyond the scope of this lecture. On the observational side, black holes are building blocks of the universe, with important roles in galaxy formation and growth regulation. The Galactic Center black hole is used for tests of general relativity, and gravitational wave detections question formation channels of stellar-mass black holes.

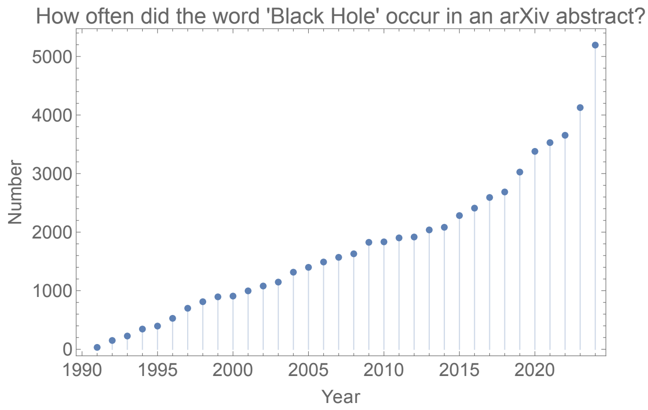


Figure 1: The number of occurrences of the word 'Black Hole' in abstracts submitted to arXiv as a function of year.

# 2 Tensor algebra - the maths of general relativity

Here is a collection of useful definitions and relations, introducing also the canonic notation for general relativity.

## 2.1 Euclidean, Cartesian coordinates

Coordinates are

$$\vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \sum_{i=1}^{3} x^i \vec{e}_i = x^i \vec{e}_i \tag{3}$$

Note that the numbers to top right of the $x$ are not "to the power of", but coordinate indices. Some care and understanding is needed, when one reads a symbol like $x^2$. For the unit vectors we have:

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \tag{4}$$

The Kronecker-$\delta$ is here the 3D unity matrix. The Euclidean dot product is

$$\begin{aligned} \vec{x} \cdot \vec{y} &= x_1 y_1 + x_2 y_2 + x_3 y_3 \\ &= \sum_{i=1}^{3} x_i y_i = x_i \, y^i \\ &= (x_1, x_2, x_3) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= \vec{x} \cdot g \cdot \vec{y} = g_{ij} \, x^i \, y^i \end{aligned} \tag{5}$$

Note the Einstein summation convention: Indices which appear twice, once upper and once lower, are automatically summed over. One can always change the name of such an index pair, as it is "dummy" (like an integration variable in an integral). This notation also gives a convenient way of checking validity of equations: Both sides need to have the same indices in upper and lower positions, after the summations are executed. Further, one has for tensors

$$A^i{}_i = g^{ij} A_{ji} = \mathrm{tr}(A^i{}_j) \tag{6}$$

The use of latin letters for indices indicates 3D, space vectors.

## 2.2 4D space-time coordinates

The coordinates are

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} \text{ or for spherical coordinates} : x^\mu = \begin{pmatrix} ct \\ r \\ \theta \\ \phi \end{pmatrix} \tag{7}$$

And with the unit vector $e_\mu$ the vector $x$ is:

$$x = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 = \sum_{\mu=0}^{3} x^\mu e_\mu = x^\mu e_\mu \tag{8}$$

For 4-vectors, we use greek indices.

### 2.2.1 Minkowski space

The dot product $x \cdot y$ is defined via

$$x \cdot y = \eta_{\mu\nu} x^\mu y^\mu \tag{9}$$

with the Minkowski metric:

$$\eta_{\mu\nu}^{\text{Cartesian}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \eta_{\mu\nu}^{\text{Spherical}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{10}$$

For a line element one has

$$\begin{aligned} ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \\ &= -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \end{aligned} \tag{11}$$

Note that there are different conventions found in the literature: The signs of the metric might be opposite, the $c^2$ could be part of the metric or in the definition of the 0-components of the coordinates (as here), or even an imaginary $i$ is used sometimes to express the opposite sign of the time component compared to the spatial components.

### 2.2.2 Curved space-time coordinates

The components of a vector are as usual

$$\begin{aligned} x &= x^\mu e_\mu \\ dx &= dx^\mu e_\mu \end{aligned} \tag{12}$$

with the novelty that the base vectors $e_\mu$ are not constants, but can be functions of the coordinates. The dot product gets generalized by going from $\eta_{\mu\nu}$ to $g_{\mu\nu}$, which can also be of a more complicated functional form. The dot product for curved manifolds is defined via

$$\begin{aligned} A(x) \cdot B(x) &= g_{\mu\nu}(x) A^\mu(x) B^\nu(x) = A_\nu(x) B^\nu(x) \\ A \cdot B &= g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu \end{aligned} \tag{13}$$

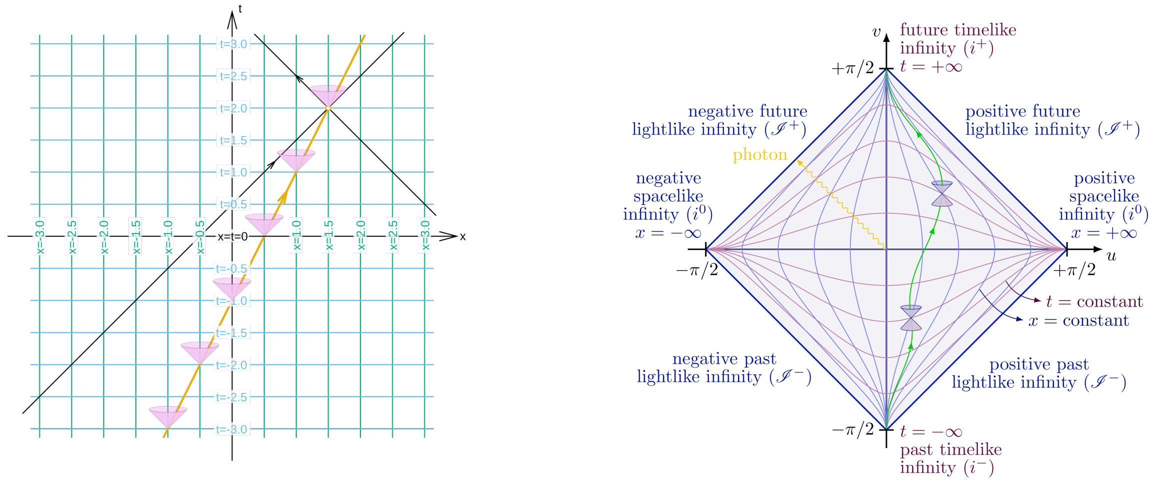


Figure 2: Two representation of a (1-1) Minkowski space-time. Left: Flat coordinates. The worldline of an observer at constant velocity is shown, together with the lightcones, giving the regions of space-time, which causally can connect to the respective event. Light rays travel diagonally, time-like trajectories are more vertical than the light rays. Right: Again Minkowski space-time, plotted in a Penrose diagram with coordinates $(u, v)$, in which are defined by $r + ct = \tan(u + v)$, $r - ct = \tan(u - v)$. This form of the diagram is useful to describe black holes later. Source: German Wikipedia and TikZ.

Note that the two vectors need to be evaluated at the same space-time point. So one can always calculate a vector length, but (in general) not the cross product of two space vectors $X, Y$ The metric tensor $g_{\mu\nu}$ is

$$
\begin{aligned}
g_{\mu\nu} &= e_\mu e_\nu = e_\nu e_\mu = g_{\nu\mu} \\
\delta^\rho_\nu &= e^{\,\rho}\, e_{\,\nu} = g^{\rho\mu} e_{\,\mu}\, e_{\,\nu} = g^{\rho\mu} g_{\mu\nu} = g^\rho_\nu \\
g_{\mu\nu} &= (g^{\mu\nu})^{-1}
\end{aligned}
\tag{14}
$$

The metric tensor is thus symmetric. The last line follows from the second line, noting that the Kronecker-$\delta$ here is the 4D unity matrix. With that one gets the line element

$$
ds^2 = (dx^\mu e_\mu)(dx^\nu e_\nu) = e_\mu e_\nu dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \tag{15}
$$

which is thus a generalization of the usual Euclidean Pythagorean theorem for infinitesimal paths in curved space-time. The coordinate transformation (changing $x \longrightarrow x'$) of a vector $A$ ($\longrightarrow A'$) is

$$
\begin{aligned}
A'^\alpha &= \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta \\
A'_\alpha &= \frac{\partial x^\beta}{\partial x'^\alpha} A_\beta
\end{aligned}
\tag{16}
$$

For the example of velocity $V$:

$$
V'^\alpha = \frac{\partial x'^\alpha}{\partial \tau} = \frac{\partial x'^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial \tau} = \frac{\partial x'^\alpha}{\partial x^\beta} V^\beta \tag{17}
$$

### 2.2.3 Note on derivatives

Derivatives are written as:

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \vec{\nabla} \end{pmatrix} \tag{18}$$

The $\partial_\mu$ is a very useful notation. In many books, one also finds the notation $X_{\mu,\nu}$ for $\partial_\nu X_\mu$. Here, we don't use it. The 4-gradient is

$$\begin{aligned} \nabla &= (\partial_0, \partial_1, \partial_2, \partial_3) \\ \nabla_{\text{Cartesian}} &= (\frac{1}{c}\partial_t, \partial_x, \partial_y, \partial_z) \\ \nabla_{\text{Spherical}} &= (\frac{1}{c}\partial_t, \partial_r, \partial_\theta, \partial_\phi) \end{aligned} \tag{19}$$

And the square of the 4-gradient (the Laplace operator) is:

$$\nabla^2 = g^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu \tag{20}$$

It requires thus the inverse of the metric tensor.

### 2.2.4 Equivalence principle

The equivalence principle states that one can gets the same experimental results in any reference frame, i.e. one can also choose a free-falling one, in which locally no gravity is felt, and hence for free falling reference systems one has $g_{\mu\nu} = \eta_{\mu\nu}$.

### 2.2.5 Example: Magnitude of 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{d}{d\tau}\vec{x} \end{pmatrix} = \begin{pmatrix} c\gamma \\ \frac{d}{dt}\vec{x}\frac{dt}{d\tau} \end{pmatrix} = \begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix} \tag{21}$$

Going to a local inertial frame, one can use the Minkowski metric.

$$\begin{aligned} |u|^2 &= u_\nu u^\nu = \eta_{\mu\nu}u^\mu u^\nu = (c\gamma, \vec{v}\gamma).\begin{pmatrix} -1 & 0 \\ \vec{0} & \mathbb{1} \end{pmatrix}.\begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix} = -c^2\gamma^2 + \vec{v}^2\gamma^2 \\ &= (v^2 - c^2)\frac{1}{1-\frac{v^2}{c^2}} = (v^2 - c^2)\frac{c^2}{c^2 - v^2} = -c^2 \end{aligned} \tag{22}$$

Since scalars are Lorentz-invariant, this result holds in any reference frame.

# 3 Relativistic dynamics

Curves are often parametrized by the "proper time" $\tau$, i.e. the time passing for a particle moving along the space-time curve. $\tau$ in general differs from the coordinate time $t$. This defines the Lorentz factor:

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{23}$$

Here, $v = |\vec{v}|$. For a free-falling observer in his local inertial frame $x' = (ct', x'^0, x'^1, x'^3) = (c\tau, \text{const}, \text{const}, \text{const})$, i.e he is at constant coordinates, and time passes at the "proper time". For him, the line elements reads thus

$$ds^2 = -c^2 d\tau^2 \tag{24}$$

which must hold in any reference frame, as it is a scalar relation. This relation holds for any real-world particle, and one calls this a "time-like" trajectory. For light, one has $ds^2 = 0$, leading to "null geodesics".

## 3.1 Energy and momentum

For particles with rest mass $m_0$ the 4-momentum $p$ is:

$$p = m_0 u = m_0 \begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} \tag{25}$$

Note that $E$ and $\vec{p}$ are different than in Newtonian mechanics. In Minkowski space, its norm is

$$|p|^2 = p_\mu p^\mu = \eta_{\nu\mu} p^\nu p^\mu = -m_0^2\gamma^2 c^2 + m_0^2\gamma^2 v^2 = -m_0^2\gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = -m_0^2\gamma^2 c^2 \frac{1}{\gamma^2} = -m_0^2 c^2 \tag{26}$$

The 4-acceleration is

$$a^\mu = \frac{d}{d\tau}\frac{p^\mu}{m_0} \tag{27}$$

4-velocity and 4-acceleration are orthogonal to each other (when $m_0$ is constant):

$$u.a = \eta_{\nu\mu}\frac{p^\nu}{m_0} a^\mu = \eta_{\nu\mu}\frac{p^\nu}{m_0}\frac{d}{d\tau}\frac{p^\mu}{m_0} = \frac{1}{2m_0^2}\frac{d}{d\tau}p^2 = \frac{1}{2m_0^2}\frac{d}{d\tau}(-m_0^2c^2) = 0 \tag{28}$$

The energy can be written also in this form:

$$E = -\eta_{\nu\mu} p^\nu u^\mu = -m_0\, \eta_{\nu\mu} u^\nu u^\mu = m_0\, c^2 \tag{29}$$

But this equation is also valid, if one measures the energy of a particle moving with $u$ in a system moving with $v$:

$$E = -\eta_{\nu\mu} p^\nu v^\mu = -m_0\, \eta_{\nu\mu} u^\nu v^\mu \tag{30}$$

This definition will be carried over to general relativity:

$$E = -g_{\nu\mu} p^\nu u^\mu \tag{31}$$

The kinetic energy $K$ is:

$$K = E - m_0 c^2 = (\gamma - 1) m_0 c^2 \tag{32}$$

Using the series expansion for $\gamma$

$$\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} \tag{33}$$

one sees that the leading order of $K$ is

$$K = (\gamma - 1) m_0 c^2 \approx \frac{1}{2} m_0 v^2 \tag{34}$$

From the definitions of $E$ and $\vec{p}$ follows:

$$\begin{aligned}
\vec{v}E/c &= \vec{p}c \\
\frac{v^2}{c^2}E^2 &= (|\vec{p}|c)^2 \\
E^2(1-\frac{v^2}{c^2}) &= E^2-(|\vec{p}|c)^2 \\
E^2/\gamma^2 &= E^2-(|\vec{p}|c)^2 \\
(m_0c^2)^2 &= E^2-(|\vec{p}|c)^2
\end{aligned} \tag{35}$$

This is the relativistic energy-momentum relation. Newton's second law takes the form

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{36}$$

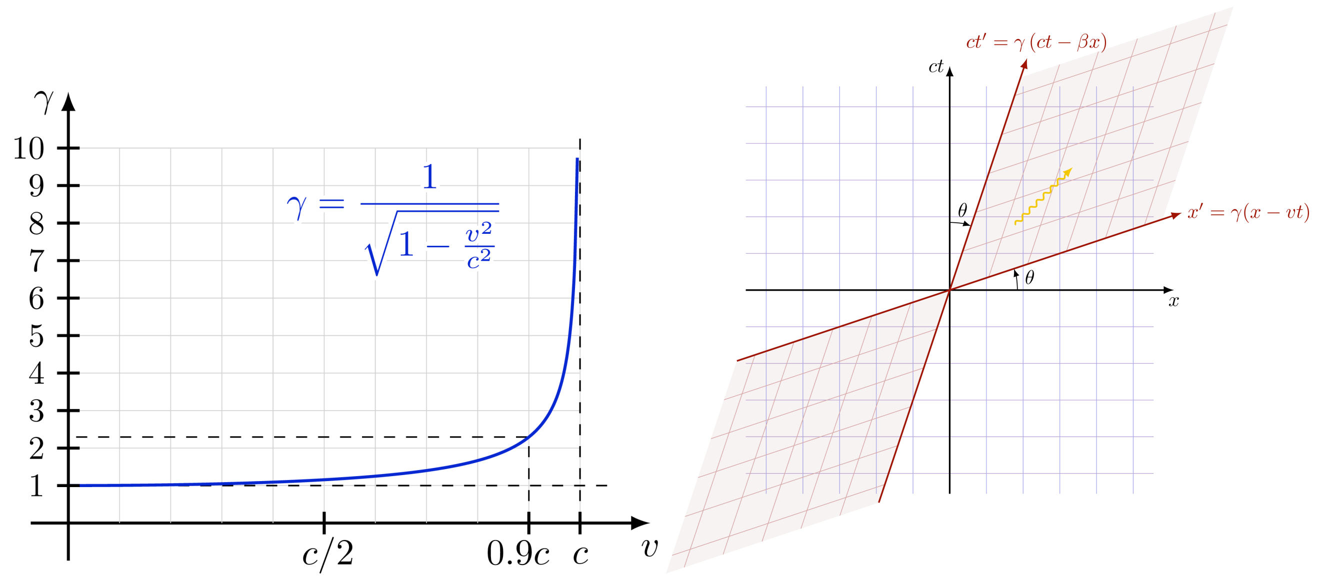


Figure 3: Left: The Lorentz-factor $\gamma$ as a function of velocity. Right: A Lorentz-transformation of the Minkowski-space time. Source: TikZ

### 3.2 Lorentz transformations

For an observer moving relative to another one along the $x$-axis with velocity $v_x$, the coordinates are

$$\begin{aligned}
t' &= \gamma\left(t-\frac{v_x}{c^2}x\right) \\
x' &= \gamma(x-v_xt) \\
y' &= y \\
z' &= z
\end{aligned} \tag{37}$$

It is easy to calculate

$$v'_x = \frac{dx'}{dt'} = \frac{d}{dt'}\gamma(x - v_x t) = \frac{d}{dt'}\gamma\left(x - v_x\left(\frac{t'}{\gamma} + \frac{v_x}{c^2}x\right)\right) = -v_x \tag{38}$$

In matrix form, one can write:

$$\begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \frac{v}{c}\gamma & 0 & 0 \\ \frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} \tag{39}$$

A space-time interval is invariant under a Lorentz transformation:

$$\begin{aligned} s'^2 &= -(c\,t')^2 + x'^2 + y'^2 + z'^2 \\ &= -c^2\gamma^2\left(t - \frac{v_x}{c^2}x\right)^2 + \gamma^2(x - v_x t)^2 + y^2 + z^2 \\ &= \gamma^2(-c^2t^2 + 2c^2\frac{v_x}{c^2}x - \frac{v_x^2}{c^2}x^2) + \gamma^2(x^2 - 2x\,v_x t + v_x^2 t^2) + y^2 + z^2 \\ &= \gamma^2c^2t^2(-1 + \frac{v_x^2}{c^2}) + \gamma^2x^2(1 - \frac{v_x^2}{c^2}) + y^2 + z^2 \\ &= \gamma^2c^2t^2\frac{-1}{\gamma^2} + \gamma^2x^2\frac{1}{\gamma^2} + y^2 + z^2 \\ &= -(c\,t)^2 + x^2 + y^2 + z^2 = s^2 \end{aligned} \tag{40}$$

The transformation law for velocities follows: The velocity $v'_x$ measured in the primed coordinate system that moves with velocity $u$ is given by the original $v_x$ and $u$ by:

$$v'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - udt)}{\gamma(dt - \frac{u}{c^2dx})} = \frac{\frac{dx}{t} - u}{1 - \frac{u}{x^2}\frac{dx}{t}} = \frac{v_x - u}{1 - \frac{u\,v_x}{c^2}} \tag{41}$$

The same transformation can be applied to the energy-momentum vector $(E/c, \vec{p})$ or the wave vector $(\omega/c, \vec{k})$. Let's take $E$ as an example:

$$\begin{aligned}
\frac{E'}{c} &= \frac{1}{c}\frac{m_0c^2}{\sqrt{1-v_x'^2/c^2}} \\
&= \frac{1}{c}\frac{m_0c^2}{\sqrt{1-\left(\frac{v_x-u}{1-\frac{u\,v_x}{c^2}}\right)^2/c^2}} \\
&= \frac{1}{c}\frac{m_0c^2\left(1-\frac{u\,v_x}{c^2}\right)}{\sqrt{\left(1-\frac{u\,v_x}{c^2}\right)^2-\frac{(v_x-u)^2}{c^2}}} \\
&= \frac{1}{c}\frac{m_0c^2\left(1-\frac{u\,v_x}{c^2}\right)}{\sqrt{1-2\frac{u\,v_x}{c^2}+\frac{u^2\,v_x^2}{c^4}-\frac{v_x^2}{c^2}+2\frac{uv_x}{c^2}-\frac{u^2}{c^2}}} \\
&= \frac{1}{c}\frac{m_0c^2\left(1-\frac{u\,v_x}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v_x^2}{c^2}\right)}} \\
&= \frac{1}{c}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(\gamma m_0c^2-u\,\gamma m_0 v_x\right) \\
&= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(\frac{E}{c}-\frac{u}{c}\,p_x\right)
\end{aligned} \tag{42}$$

So, this is the same as if we would have transformed $(E/c, \vec{p})$ by the same Lorentz boost by $u$ as we did for $x$.

# 4 Curvature

## 4.1 Describing curvature

### 4.1.1 Covariant derivative and Christoffel symbols

"If I move a vector into the direction of another vector, how do its components change" - a non-trivial question in curved space-time. For a base vector $e_\mu$ moving infinitesimal into direction $x^\nu$, the four components for the $e_\lambda$ are given by the Christoffel symbols (which are not tensors):

$$\begin{aligned}
\Gamma^\lambda{}_{\mu\nu} e_\lambda &= \frac{\partial e_\mu}{\partial x^\nu} \\
\Gamma^\lambda{}_{\mu\nu} e_\lambda e^\beta &= \frac{\partial e_\mu}{\partial x^\nu} e^\beta \\
\Gamma^\lambda{}_{\mu\nu} \delta^\beta_\lambda &= \frac{\partial e_\mu}{\partial x^\nu} e^\beta \\
\Gamma^\beta{}_{\mu\nu} &= \frac{\partial e_\mu}{\partial x^\nu} e^\beta
\end{aligned} \tag{43}$$

This allows defining a covariant derivative: "The vector field $A$ not only changes as a function of coordinates, but due to the curvature, there is also a change due to the coordinates changing."

$$\begin{aligned}\nabla_\mu A^\lambda &= \partial_\mu A^\lambda + \Gamma^\lambda_{\ \mu\nu} A^\nu \\ \nabla_\mu A_\lambda &= \partial_\mu A_\lambda - \Gamma^\nu_{\ \mu\lambda} A_\nu \end{aligned} \tag{44}$$

We will not use the notation $A^\lambda_{\ ;\mu} = \nabla_\mu A^\lambda$. For a tensor, one has:

$$\nabla_\mu A^{\alpha\beta} = \partial_\mu A^{\alpha\beta} + \Gamma^\alpha_{\ \lambda\mu} A^{\lambda\beta} + \Gamma^\beta_{\ \mu\lambda} A^{\alpha\lambda} \tag{45}$$

And for a scalar $f$, the covariant derivative is the partial one:

$$\nabla_\lambda f = \partial_\lambda f \tag{46}$$

The minus sign in equation 44 appears because the following line should be true (i.e. the covariant derivative should behave like a derivative):

$$(\partial_\lambda A^\mu) B_\mu + A^\mu(\partial_\lambda B_\mu) = \partial_\lambda(A^\mu B_\mu) = \nabla_\lambda(A^\mu B_\mu) = (\nabla_\lambda A^\mu) B_\mu + A^\mu(\nabla_\lambda B_\mu) \tag{47}$$

and the two terms with Christoffel symbols in the covariant derivatives need therefore to cancel. Further, one sees that in that was also the chain rule for the product of two vectors holds. One can show the chain rule also for higher ranked tensors, for example:

$$\begin{aligned}\nabla_\mu(A^{\alpha\beta}B_\gamma) &= \partial_\mu(A^{\alpha\beta}B_\gamma) + \Gamma^\alpha_{\ \lambda\mu} A^{\lambda\beta} B_\gamma + \Gamma^\beta_{\ \mu\lambda} A^{\alpha\lambda} B_\gamma - \Gamma^\lambda_{\ \gamma\mu} A^{\alpha\beta} B_\lambda \\ &= B_\gamma \partial_\mu A^{\alpha\beta} + A^{\alpha\beta}\partial_\mu B_\gamma + B_\gamma \Gamma^\alpha_{\ \lambda\mu} A^{\lambda\beta} + B_\gamma \Gamma^\beta_{\ \mu\lambda} A^{\alpha\lambda} - A^{\alpha\beta} B_\lambda \Gamma^\lambda_{\ \gamma\mu} \\ &= A^{\alpha\beta}(\partial_\mu B_\gamma - B_\lambda \Gamma^\lambda_{\ \gamma\mu}) + B_\gamma(\partial_\mu A^{\alpha\beta} + \Gamma^\alpha_{\ \lambda\mu} A^{\lambda\beta} + \Gamma^\beta_{\ \mu\lambda} A^{\alpha\lambda}) \\ &= A^{\alpha\beta}\nabla_\mu B_\gamma + B_\gamma \nabla_\mu A^{\alpha\beta}\end{aligned} \tag{48}$$

With the chosen definition for the covariant derivative, a property called metric compatibility holds: The covariant derivative of the metric tensor is always zero:

$$\begin{aligned}\nabla_\lambda g^{\alpha\mu} &= 0 \\ \nabla_\lambda g_{\alpha\mu} &= 0\end{aligned} \tag{49}$$

This allows moving the metric tensor in and out of derivatives:

$$\begin{aligned}\nabla_\lambda A^\nu &= \nabla_\lambda(g^{\nu\mu}A_\mu) = A_\mu \nabla_\lambda g^{\nu\mu} + g^{\nu\mu}\nabla_\lambda A_\mu = 0 + g^{\nu\mu}\nabla_\lambda A_\mu = g^{\nu\mu}\nabla_\lambda A_\mu \\ \nabla_\lambda A_\nu &= \nabla_\lambda(g_{\nu\mu}A^\mu) = A^\mu \nabla_\lambda g_{\nu\mu} + g_{\nu\mu}\nabla_\lambda A^\mu = 0 + g_{\nu\mu}\nabla_\lambda A^\mu = g_{\nu\mu}\nabla_\lambda A^\mu\end{aligned} \tag{50}$$

For a torsion-free space-time, we expect that derivatives commute: $\nabla_\mu\nabla_\nu = \nabla_\nu\nabla_\mu$. For a scalar $f$ in flat coordinates the covariant derivative is the partial derivative, and clearly

$$\partial_\mu\partial_\nu f = \partial_\nu\partial_\mu f \tag{51}$$

If the symmetry holds in one coordinate system, it holds in all, hence

$$\begin{aligned}\nabla_\mu\partial_\nu f &= \nabla_\nu\partial_\mu f \\ \partial_\mu\partial_\nu f - \Gamma^\alpha_{\ \mu\nu}\partial_\alpha f &= \partial_\nu\partial_\mu f - \Gamma^\alpha_{\ \nu\mu}\partial_\alpha f\end{aligned} \tag{52}$$

and we see that the metric tensor needs to be symmetric in the second and third index.

$$\Gamma^{\alpha}_{\ \mu\nu} = \Gamma^{\alpha}_{\ \nu\mu} \tag{53}$$

There are thus not $4^3 = 64$ independent Christoffel symbols, but only 40. The Christoffel symbols can be expressed with the metric tensor. This requires some algebra. From the definition we have (equation 43):

$$\begin{aligned} e_\lambda \Gamma^{\lambda}_{\ \alpha\nu} &= e_\lambda \Gamma^{\lambda}_{\ \nu\alpha} = \frac{\partial e_\nu}{\partial x^\alpha} = \partial_\alpha e_\nu \\ e_\mu e_\lambda \Gamma^{\lambda}_{\ \alpha\nu} &= e_\mu \partial_\alpha e_\nu \end{aligned} \tag{54}$$

Consider

$$\partial_\alpha (e_\mu e_\nu) = e_\nu \partial_\alpha e_\mu + e_\mu \partial_\alpha e_\nu \quad \longrightarrow \quad e_\mu \partial_\alpha e_\nu = \partial_\alpha (e_\mu e_\nu) - e_\nu \partial_\alpha e_\mu \tag{55}$$

With that we get:

$$e_\mu e_\lambda \Gamma^{\lambda}_{\ \alpha\nu} = \partial_\alpha (e_\mu e_\nu) - e_\nu e_\lambda \Gamma^{\lambda}_{\ \alpha\mu} \tag{56}$$

Writing this equation two more times, but with indices relabelled:

$$e_\alpha e_\lambda \Gamma^{\lambda}_{\ \nu\mu} = \partial_\nu (e_\alpha e_\mu) - e_\mu e_\lambda \Gamma^{\lambda}_{\ \nu\alpha} \tag{57}$$

$$e_\alpha e_\lambda \Gamma^{\lambda}_{\ \mu\nu} = \partial_\mu (e_\alpha e_\nu) - e_\nu e_\lambda \Gamma^{\lambda}_{\ \mu\alpha} \tag{58}$$

Taking 56 + 57 - 58 yields, using $\Gamma^{\lambda}_{\ \nu\mu} = \Gamma^{\lambda}_{\ \mu\nu}$ and $\Gamma^{\lambda}_{\ \alpha\mu} = \Gamma^{\lambda}_{\ \mu\alpha}$:

$$\begin{aligned} e_\mu e_\lambda \Gamma^{\lambda}_{\ \alpha\nu} &= \partial_\alpha (e_\mu e_\nu) + \partial_\nu (e_\alpha e_\mu) - \partial_\mu (e_\alpha e_\nu) - e_\mu e_\lambda \Gamma^{\lambda}_{\ \nu\alpha} \\ 2 e_\mu e_\lambda \Gamma^{\lambda}_{\ \alpha\nu} &= \partial_\alpha (e_\mu e_\nu) + \partial_\nu (e_\alpha e_\mu) - \partial_\mu (e_\alpha e_\nu) \\ (e^\mu e^\rho)(e_\mu e_\lambda) \Gamma^{\lambda}_{\ \alpha\nu} &= \frac{1}{2} (e^\mu e^\rho) (\partial_\alpha (e_\mu e_\nu) + \partial_\nu (e_\alpha e_\mu) - \partial_\mu (e_\alpha e_\nu)) \\ \delta^\rho_\lambda \Gamma^{\lambda}_{\ \alpha\nu} = \Gamma^{\rho}_{\ \alpha\nu} &= \frac{1}{2} g^{\mu\rho} (\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu}) \\ \Gamma^{\lambda}_{\ \mu\nu} &= \frac{1}{2} g^{\lambda\alpha} (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \\ \Gamma_{\lambda\mu\nu} &= \frac{1}{2} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \end{aligned} \tag{59}$$

With this, we can proof the metric compatibility in few steps:

$$\begin{aligned} \nabla_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - \Gamma^{\alpha}_{\ \lambda\mu} g_{\alpha\nu} - \Gamma^{\alpha}_{\ \lambda\nu} g_{\mu\alpha} = \partial_\lambda g_{\mu\nu} - \Gamma_{\nu\lambda\mu} - \Gamma_{\mu\lambda\nu} \\ &= \partial_\lambda g_{\mu\nu} - \frac{1}{2} (\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu}) - \frac{1}{2} (\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\lambda\nu}) = 0 \end{aligned} \tag{60}$$

#### 4.1.2 Riemann tensor, Ricci tensor and Ricci scalar

The idea behind the description of curvature is that parallel shifting a vector in coordinate direction $\mu$ and then in $\nu$ does not yield the same as first in direction $\nu$ and then in $\mu$, see figure 4. Expressed in infinitesimal steps, the

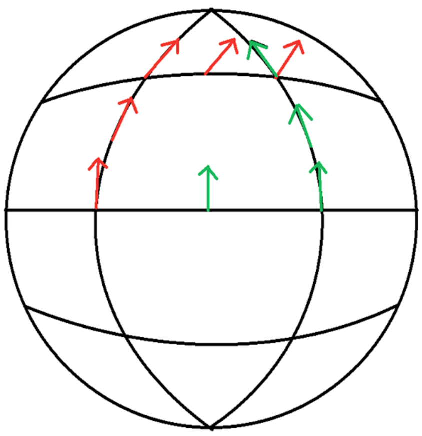

Figure 4: Left: Parallel shifting a vector from bottom left to top right yields different results, depending on the along which coordinate axis one moves first and which second. This is feature of the curvature of the underlying space. Source: medium.com

difference for a vector $A^\rho$ is:

$$\begin{aligned}
(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)A^\rho &= \nabla_\mu(\partial_\nu A^\rho + \Gamma^\rho{}_{\nu\sigma}A^\sigma) - \nabla_\nu(\partial_\mu A^\rho + \Gamma^\rho{}_{\mu\sigma}A^\sigma) \\
&= \partial_\mu(\partial_\nu A^\rho + \Gamma^\rho{}_{\nu\sigma}A^\sigma) - \Gamma^\lambda{}_{\mu\nu}(\partial_\lambda A^\rho + \Gamma^\rho{}_{\lambda\sigma}A^\sigma) + \Gamma^\rho{}_{\mu\lambda}(\partial_\nu A^\lambda + \Gamma^\lambda{}_{\nu\sigma}A^\sigma) \\
&\quad -\partial_\nu(\partial_\mu A^\rho + \Gamma^\rho{}_{\mu\sigma}A^\sigma) + \Gamma^\lambda{}_{\nu\mu}(\partial_\lambda A^\rho + \Gamma^\rho{}_{\lambda\sigma}A^\sigma) - \Gamma^\rho{}_{\nu\lambda}(\partial_\mu A^\lambda + \Gamma^\lambda{}_{\mu\sigma}A^\sigma) \\
&= (\partial_\mu\Gamma^\rho{}_{\nu\sigma})A^\sigma + \Gamma^\rho{}_{\nu\sigma}\partial_\mu A^\sigma - \Gamma^\lambda{}_{\mu\nu}\partial_\lambda A^\rho - \Gamma^\lambda{}_{\mu\nu}\Gamma^\rho{}_{\lambda\sigma}A^\sigma + \Gamma^\rho{}_{\mu\lambda}\partial_\nu A^\lambda + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma}A^\sigma \\
&\quad -(\partial_\nu\Gamma^\rho{}_{\mu\sigma})A^\sigma - \Gamma^\rho{}_{\mu\sigma}\partial_\nu A^\sigma + \Gamma^\lambda{}_{\nu\mu}\partial_\lambda A^\rho + \Gamma^\lambda{}_{\nu\mu}\Gamma^\rho{}_{\lambda\sigma}A^\sigma - \Gamma^\rho{}_{\nu\lambda}\partial_\mu A^\lambda - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}A^\sigma \quad (61)
\end{aligned}$$

The third terms in both rows cancel, as well as the fourth terms. Term 5 in row 1 after relabelling the summation index $\lambda$ into $\sigma$ cancels term 2 in row 2. Term 2 in row 1 after relabelling the summation index $\sigma$ into $\lambda$ cancels term 5 in row 2. We are left with

$$... \quad = \quad (\partial_\mu\Gamma^\rho{}_{\mu\sigma} - \partial_\nu\Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma})A^\sigma \quad (62)$$

The term in brackets vanishes, if the parallel shifting of the vector happens in flat space. In curved space-time it does not, and it measures the curvature. It is called the Riemann curvature tensor:

$$R^\lambda{}_{\alpha\nu\mu} = \partial_\nu\Gamma^\lambda{}_{\alpha\mu} - \partial_\mu\Gamma^\lambda{}_{\alpha\nu} + \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\alpha\mu} - \Gamma^\lambda{}_{\sigma\mu}\Gamma^\sigma{}_{\alpha\nu} \quad (63)$$

Here it is defined with the "Riemann" sign convention. It has the following symmetries:

$$\begin{aligned}
R_{\lambda\alpha\nu\mu} &= -R_{\alpha\lambda\mu\nu} \\
R^\lambda{}_{\alpha\nu\mu} &= -R^\lambda{}_{\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \quad (64)
\end{aligned}$$

One obtains the so-called Ricci tensor as contraction from the Riemann tensor, or expressed in terms of Christoffel symbols:

$$\begin{aligned} R_{\alpha\mu} &= R^{\lambda}{}_{\alpha\lambda\mu} \\ &= \partial_\lambda \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\mu \Gamma^{\lambda}{}_{\alpha\lambda} + \Gamma^{\lambda}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\alpha\mu} - \Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\alpha\lambda} \end{aligned} \tag{65}$$

The Ricci tensor is symmetric

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} = g^{\beta\alpha} R_{\alpha\nu\beta\mu} = R^{\beta}_{\nu\beta\mu} = R_{\nu\mu} \tag{66}$$

The meaning of the Ricci tensor is that it describes the volume changes as a function of coordinates of an infinitesimal space-time element. The Ricci scalar is a contraction of the Ricci tensor:

$$R = R^{\alpha}_{\alpha} \tag{67}$$

## 4.2 Deriving the Bianchi identities

Working for the moment in a special coordinate system - the result will be a scalar, and hence independent of the choice of coordinate system. Going to a local inertial frame. Then, locally the metric is Minkowski, and the Christoffel symbols vanish. But: not the derivatives of the Christoffel symbols! "One can transform away locally the gravitational force, but not the tidal forces". Also, in a local inertial frame, the the covariant derivative becomes a partial derivative. The Riemann curvature in a local inertial frame is:

$$R^{\lambda}{}_{\alpha\nu\mu} = \partial_\nu \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\mu \Gamma^{\lambda}{}_{\alpha\nu} \tag{68}$$

Taking the covariant derivative:

$$\nabla_\sigma R^{\lambda}{}_{\alpha\nu\mu} = \partial_\sigma \partial_\nu \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\sigma \partial_\mu \Gamma^{\lambda}{}_{\alpha\nu} \tag{69}$$

Writing this three times, cycling indices:

$$\begin{aligned} \nabla_\sigma R^{\lambda}{}_{\alpha\nu\mu} &= \partial_\sigma \partial_\nu \Gamma^{\lambda}{}_{\alpha\mu} - \partial_\sigma \partial_\mu \Gamma^{\lambda}{}_{\alpha\nu} \\ \nabla_\nu R^{\lambda}{}_{\alpha\mu\sigma} &= \partial_\nu \partial_\mu \Gamma^{\lambda}{}_{\alpha\sigma} - \partial_\nu \partial_\sigma \Gamma^{\lambda}{}_{\alpha\mu} \\ \nabla_\mu R^{\lambda}{}_{\alpha\sigma\nu} &= \partial_\mu \partial_\sigma \Gamma^{\lambda}{}_{\alpha\nu} - \partial_\mu \partial_\nu \Gamma^{\lambda}{}_{\alpha\sigma} \end{aligned} \tag{70}$$

Adding the three equations, and noting that partial derivatives commute, the right hand sides yields simply 0, which is known as the Bianchi identity

$$\nabla_\sigma R^{\lambda}{}_{\alpha\nu\mu} + \nabla_\nu R^{\lambda}{}_{\alpha\mu\sigma} + \nabla_\mu R^{\lambda}{}_{\alpha\sigma\nu} = 0 \tag{71}$$

This result is valid is any reference frame, as it is a tensor relation.

# 5 Geodesic equation

## 5.1 Standard approach

A geodesic is the equivalent of a straight line in Euclidean or Minkowski space. A body with no acceleration will move along such a line, and it is defined by velocity being constant:

$$0 = \frac{d\vec{u}}{d\tau} = \frac{d(u^\mu \vec{e}_\mu)}{d\tau} = \vec{e}_\mu \frac{du^\mu}{d\tau} + u^\mu \frac{d\vec{e}_\mu}{d\tau} \tag{72}$$

From this the geodesic equation can be derived:

$$\begin{aligned}
\vec{e}_\mu \frac{du^\mu}{d\tau} &= -u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
\vec{e}_\nu \vec{e}_\mu \frac{du^\mu}{d\tau} &= -\vec{e}_\nu u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
g_{\mu\nu} \frac{du^\mu}{d\tau} &= -\vec{e}_\nu u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
\frac{du_\nu}{d\tau} &= -\vec{e}_\nu u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
g^{\nu\lambda} \frac{du_\nu}{d\tau} &= -g^{\nu\lambda}\, \vec{e}_\nu\, u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
\frac{du^\lambda}{d\tau} &= -\vec{e}^{\,\lambda}\, u^\mu \frac{d\vec{e}_\mu}{d\tau} \\
\frac{du^\lambda}{d\tau} &= -\vec{e}^{\,\lambda}\, u^\mu \frac{d\vec{e}_\mu}{dx^\nu} \frac{dx^\nu}{d\tau} \\
\frac{du^\lambda}{d\tau} &= -\frac{\partial \vec{e}_\mu}{\partial x^\nu}\, \vec{e}^{\,\lambda}\, u^\mu u^\nu \\
\frac{du^\lambda}{d\tau} &= -\Gamma^\lambda{}_{\mu\nu} u^\mu u^\nu
\end{aligned} \tag{73}$$

Hence, we have the two equivalent forms:

$$\begin{aligned}
\frac{du^\lambda}{d\tau} + \Gamma^\lambda{}_{\mu\nu} u^\mu u^\nu &= 0 \\
\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} &= 0
\end{aligned} \tag{74}$$

It is written here for proper time $\tau$, but actually holds for any parameter.

## 5.2 Lagrangian formulation

There is an elegant way to express the geodesic equation using the Lagrangian $\mathcal{L}$.

$$\mathcal{L}(x,u) = -mc\sqrt{-g_{\mu\nu}(x) u^\mu u^\nu} \tag{75}$$

Note that formally $\mathcal{L} = -m\,c^2$. Geodesics are extremal in the sense, that they are the shortest paths between two points. The $\sqrt{X}$ is extremal when $X$ is extremal, and hence one can also vary

$$\mathcal{L}'(x,u) = g_{\mu\nu}(x) u^\mu u^\nu \tag{76}$$

As usual, the coordinates $x$ and velocities $u$ are treated as independent variables. The Euler-Lagrange equation is then

$$\frac{\partial \mathcal{L}'}{\partial x^\lambda} = \frac{d}{d\tau} \frac{\partial \mathcal{L}'}{\partial u^\lambda} \tag{77}$$

This is identical to the geodesic equation. The left side is

$$\frac{\partial \mathcal{L}'}{\partial x^\lambda} = \frac{1}{2} (\partial_\lambda g_{\mu\nu}) u^\mu u^\nu \tag{78}$$

The right side is

$$\begin{aligned}
\frac{d}{d\tau}\frac{\partial \mathcal{L}'}{\partial u^\lambda} &= \frac{d}{d\tau}\left(\frac{1}{2}g_{\mu\nu}\delta^\mu_\lambda u^\nu + \frac{1}{2}g_{\mu\nu}u^\mu\delta^\nu_\lambda\right) \\
&= \frac{d}{d\tau}\left(g_{\mu\lambda}u^\mu\right) \\
&= \partial_\nu g_{\mu\lambda}u^\nu u^\mu + g_{\mu\lambda}\ddot{x}^\mu \\
&= \frac{1}{2}\partial_\nu g_{\mu\lambda}u^\nu u^\mu + \frac{1}{2}\partial_\mu g_{\nu\lambda}u^\mu u^\nu + g_{\mu\lambda}\ddot{x}^\mu
\end{aligned} \tag{79}$$

In the last step, the previous term was split and summation indices have been renamed. The dot-derivative indicates deriving with respect to proper time. Putting things together

$$\begin{aligned}
\frac{1}{2}\partial_\lambda g_{\mu\nu}u^\mu u^\nu &= \frac{1}{2}\partial_\nu g_{\mu\lambda}u^\nu u^\mu + \frac{1}{2}\partial_\mu g_{\nu\lambda}u^\mu u^\nu + g_{\mu\lambda}\ddot{x}^\mu \\
0 &= g_{\mu\lambda}\ddot{x}^\mu + \frac{1}{2}\partial_\nu g_{\mu\lambda}u^\nu u^\mu + \frac{1}{2}\partial_\mu g_{\nu\lambda}u^\mu u^\nu - \frac{1}{2}\partial_\lambda g_{\mu\nu}u^\mu u^\nu \\
0 &= \ddot{x}_\lambda + \Gamma_{\lambda\mu\nu}u^\mu u^\nu \\
0 &= g^{\lambda\rho}\ddot{x}_\lambda + g^{\lambda\rho}\Gamma_{\lambda\mu\nu}u^\mu u^\nu \\
0 &= \ddot{x}^\rho + \Gamma^\rho{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu
\end{aligned} \tag{80}$$

An advantage of this formulation is that often it avoids using Christoffel symbols.

### 5.3 The choice of Lagrangian

Since we optimized $\mathcal{L}'$, why don't we call that Lagrangian? The reason is, that the action

$$S_\tau = \int d\tau \mathcal{L} \tag{81}$$

shall be invariant under transformations from one affine parameter $\tau$ to another one $\sigma$:

$$\begin{aligned}
S_\sigma &= -mc\int d\sigma\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}} \\
&= -mc\int \frac{d\sigma}{d\tau}d\tau\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{d\tau^2}{d\sigma^2}} \\
&= -mc\int d\tau\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = S_\tau
\end{aligned} \tag{82}$$

This does not hold for $\mathcal{L}'$, but the choice of $\mathcal{L}$ in equation 75 is invariant under re-parametrization.

## 6 Energy-momentum tensor

Recap of continuity equation for charge (expressing conservation of the scalar charge):

$$\frac{d\rho}{dt} + \vec{\nabla}\vec{j} = 0 \tag{83}$$

Using the 4-current $j^\mu = (c\rho, \vec{j})$ the continuity equation gets

$$\partial_\mu j^\mu = 0 \tag{84}$$

Conserving a scalar quantity is thus an equation with a gradient of a 4-vector. The four-momentum is

$$p^\mu = \begin{pmatrix} \frac{E}{c} \\ \gamma m_0 \vec{v} \end{pmatrix} \tag{85}$$

Conserving energy and momentum, i.e. the four-momentum, being a vector, is thus an equation with a gradient of some $4x4$ tensor:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{86}$$

Symbolically, the components (in a local inertial frame) are:

$$T = \partial_V \begin{pmatrix} E & p & p & p \\ p & P & S & S \\ p & S & P & P \\ p & S & S & P \end{pmatrix} \tag{87}$$

Note the $T_{00}$ component: It is the energy density, which is the relativistic version of mass density (as in Newtonian mechanics), and all energy components need to be taken into account. So, for the Newtonian limit, $T_{00} = \rho c^2$, which also holds for pressure-less dust. $p$ refers to momentum, $P$ to pressure, and $S$ to shear. The form of T for various cases can easily be looked up, for example for a perfect fluid.

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu + P g^{\mu\nu} \tag{88}$$

In a local inertial frame, this is a diagonal matrix with diagonal $(\rho c^2, P, P, P)$.

# 7 Derivation of the field equations

## 7.1 Finding the right tensors

- Independence of coordinate system choice: Tensor equations
- Spacetime curvature should relate to matter
- In a weak-field, slow motion limit, one should recover Newton's equations
- locally, energy and momentum should be conserved

We thus look for a tensor equation. The simplest form in which curvature can occur is a two-index curvature tensor. Matter, in the simplest form, can then be represented by the energy-momentum tensor $T^{\mu\nu}$. So we postulate some equation of type $G^{\mu\nu} \propto T^{\mu\nu}$, with an unknown tensor G. As energy and momentum shall be conserved, i.e. $\nabla_\mu T^{\mu\nu} = 0$, one also has to demand that $\nabla_\mu G^{\mu\nu} = 0$. And we can construct one, starting from the Bianchi identity:

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \tag{89}$$

Multiplying with $g^{\alpha\mu}$ and swapping indices in the second term:

$$g^{\alpha\mu}\nabla_\lambda R_{\alpha\beta\mu\nu} - g^{\alpha\mu}\nabla_\nu R_{\alpha\beta\mu\lambda} + g^{\alpha\mu}\nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \tag{90}$$

Due to the metric compatibility, one can move the $g^{\alpha\mu}$ into the derivatives, where it will raise the $\alpha$ index to $\mu$ (first two terms) and:

$$\nabla_\lambda R^\mu{}_{\beta\mu\nu} - \nabla_\nu R^\mu{}_{\beta\mu\lambda} + \nabla^\alpha R_{\alpha\beta\nu\lambda} = 0 \tag{91}$$

In the first and second term, the Ricci tensor appeared, and swapping indices in the last term:

$$\nabla_\lambda R_{\beta\nu} - \nabla_\nu R_{\beta\lambda} - \nabla^\alpha R_{\beta\alpha\nu\lambda} = 0 \tag{92}$$

Multiplying with $g^{\beta\nu}$ and again using the metric compatibility, raising indices, and substituting Ricci tensor and scalar when they occur, one gets

$$\begin{aligned}
\nabla_\lambda g^{\beta\nu} R_{\beta\nu} - \nabla_\nu g^{\beta\nu} R_{\beta\lambda} - \nabla^\alpha g^{\beta\nu} R_{\beta\alpha\nu\lambda} &= 0 \\
\nabla_\lambda R^\nu_\nu - \nabla_\nu R^\nu_\lambda - \nabla^\alpha R^\nu_{\alpha\nu\lambda} &= 0 \\
\nabla_\lambda R - \nabla_\nu R^\nu_\lambda - \nabla^\alpha R_{\alpha\lambda} &= 0
\end{aligned} \tag{93}$$

Using $\nabla^\alpha = g^{\alpha\rho}\nabla_\rho$ and using again the metric compatibility:

$$\begin{aligned}
\nabla_\lambda R - \nabla_\nu R^\nu_\lambda - \nabla_\rho g^{\alpha\rho} R_{\alpha\lambda} &= 0 \\
\nabla_\lambda R - \nabla_\nu R^\nu_\lambda - \nabla_\rho R^\rho_\lambda &= 0
\end{aligned} \tag{94}$$

Here, $\nu$ and $\rho$ are just summation indices, so the second and third term are the same.

$$\nabla_\lambda R - 2\nabla_\nu R^\nu_\lambda = 0 \tag{95}$$

Multiplying with $g^{\mu\lambda}$, and again using the metric compatibility and raising indices:

$$\begin{aligned}
\nabla_\lambda g^{\mu\lambda} R - 2\nabla_\nu g^{\mu\lambda} R^\nu_\lambda &= 0 \\
\nabla_\lambda g^{\mu\lambda} R - 2\nabla_\nu R^{\mu\nu} &= 0 \\
\nabla_\nu g^{\mu\nu} R - 2\nabla_\nu R^{\mu\nu} &= 0 \\
2\nabla_\nu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) &= 0
\end{aligned} \tag{96}$$

Define $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$, the last equation states $\nabla_\mu G^{\mu\nu} = 0$. Hence, G is a divergence free, two-index curvature tensor - so it is a viable candidate for the field equation. It is the simple-most such choice. Therefore, we can guess here, that the field equations are

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu} \tag{97}$$

## 7.2 Weak fields and slow motions

Note that there is no way to prove the field equations, as they are the theory Einstein **postulated**. But one can connect them to classical mechanics for the limit of a weak field (i.e. slow motions), which can serve at least as a motivation, as it shows that in this limit, we recover what we know.

### 7.2.1 Connection to Newtonian potential

First, it is useful to see how the Einstein tensor $G^{\mu\nu}$ connects to the Newton potential in the case of a weak field. For a weak field, the metric should be almost Minkowski, such that one can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{98}$$

with a small $h \ll 1$. For this weak-field metric, the Christoffel symbols are

$$\begin{aligned}
\Gamma^{\lambda}{}_{\mu\nu} &= \frac{1}{2}(\eta^{\lambda\alpha} + h^{\lambda\alpha})\left[\partial_\mu(\eta_{\alpha\nu} + h_{\alpha\nu}) + \partial_\nu(\eta_{\alpha\mu} + h_{\alpha\mu}) - \partial_\alpha(\eta_{\mu\nu} + \epsilon h_{\mu\nu})\right] \\
&= \frac{1}{2}\eta^{\lambda\alpha}(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}) \\
&= \frac{1}{2}(\partial_\mu h^{\lambda}{}_{\nu} + \partial_\nu h^{\lambda}{}_{\mu} - \partial^{\lambda} h_{\mu\nu})
\end{aligned} \tag{99}$$

where higher order terms of $h$ are dropped, and the derivatives of the constant Minkowski metric vanish. The Newtonian equations only have spatial derivatives, and thus only $i = 1, 2, 3$ of the 00-component matter:

$$\Gamma^{i}{}_{00} = \frac{1}{2}(\partial_0 h^{i}{}_{0} + \partial_0 h^{i}{}_{0} - \partial^{i} h_{00}) \tag{100}$$

The time derivatives $\partial_0$ are 0, as our metric is constant. It remains the spatial one, and we get

$$\Gamma^{i}{}_{00} \approx -\frac{1}{2}\partial^{i} h_{00} \tag{101}$$

Note that since $h$ is small, so is $\Gamma^{i}{}_{00}$. Next, one evaluates the (spatial part of the) geodesic equation. We are dealing with slow motions, $v \ll c$. In that limit, $\tau \to t$, and hence

$$\frac{d^2x^i}{dt^2} + \Gamma^{i}{}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 \tag{102}$$

The velocity terms $\frac{dx^i}{dt}$ are much smaller than $\frac{dx^0}{dt} = c$ and thus

$$\begin{aligned}
\frac{d^2x^i}{dt^2} + \Gamma^{i}{}_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} &= 0 \\
\frac{d^2x^i}{dt^2} + c^2\Gamma^{i}{}_{00} &= 0
\end{aligned} \tag{103}$$

As expected, only the $\Gamma^{i}{}_{00}$ Christoffel symbol appears. Using the calculation from above

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2\partial^{i} h_{00} = \frac{1}{2}c^2\vec{\nabla}^{i} h_{00} \tag{104}$$

In Newton's theory we have

$$\frac{d^2x^i}{dt^2} = -\vec{\nabla}^{i}\Phi \tag{105}$$

Therefore, we can identify

$$\begin{aligned}
h_{00} &= -\frac{2\Phi}{c^2} \\
g_{00} &= -1 - \frac{2\Phi}{c^2}
\end{aligned} \tag{106}$$

### 7.2.2 Proportionality constant

In Newtonian gravity, the only source of gravity is mass or energy density, which corresponds to the $T_{00}$-component of the energy-momentum tensor, and all other components being 0:

$$T_{00} = \rho c^2 \tag{107}$$

Hence, we only need to look at the 00-component for determining the proportionality constant $\kappa$.

$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = \kappa \rho c^2 \tag{108}$$

Using the Ricci tensor in the form expressed as Christoffel symbols, one sees that in the weak field limit, the third and fourth term can be neglected, as they are squares of $\Gamma$, with $\Gamma \sim h \ll 1$. Hence one has

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} \tag{109}$$

The 00-component is

$$R_{00} = \partial_\alpha \Gamma^\alpha{}_{00} - \partial_0 \Gamma^\alpha{}_{0\alpha} \tag{110}$$

As our metric tensor is not time variable, $\partial_0$ yields 0. Thus

$$R_{00} = \partial_i \Gamma^i{}_{00} - \partial_0 \Gamma^0{}_{00} \tag{111}$$

where the last term again is 0. Then

$$R_{00} = \partial_i \left( -\frac{1}{2} \partial^i h_{00} \right) = -\frac{1}{2} \partial_i \partial^i \left( -\frac{2\Phi}{c^2} \right) = \frac{1}{c^2} \vec{\nabla}^2 \Phi \tag{112}$$

Given that only the $T_{00}$ component is present in the Newtonian limit, the spatial components $G_{ij}$ are 0 - which allows for a little trick to evaluate the Ricci scalar:

$$R_{ij} - \frac{1}{2} g_{ij} R = 0 \longrightarrow R_{ij} = \frac{1}{2} g_{ij} R \tag{113}$$

Splitting up the contraction sum of the Ricci scalar:

$$R = R^\nu_\nu = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{ij} R_{ij} = g^{00} R_{00} + \frac{1}{2} g^{ij} g_{ij} R = g^{00} R_{00} + \frac{3}{2} R \tag{114}$$

where the identity $g^{ij} g_{ij} = 3$ has been used. This can be solved for $R$ (end of trick):

$$R = -2 g^{00} R_{00} \tag{115}$$

With $g^{00} = 1/g_{00}$ the Ricci tensor is

$$R = -2 \left( \frac{1}{-1 - \frac{2\Phi}{c^2}} \right) \frac{1}{c^2} \vec{\nabla}^2 \Phi \approx 2 \left( 1 + \frac{2\Phi}{c^2} \right) \frac{1}{c^2} \vec{\nabla}^2 \Phi \approx \frac{2}{c^2} \vec{\nabla}^2 \Phi + O(\Phi^2) \tag{116}$$

We thus have $R$ and $R_{00}$, and can evaluate the 00-component of the field equation:

$$\begin{aligned}
R_{00} - \frac{1}{2} g_{00} R &= \kappa T_{00} \\
\frac{1}{c^2}\vec{\nabla}^2\Phi - \frac{1}{2}\left(-1 - \frac{2\Phi}{c^2}\right)\frac{2}{c^2}\vec{\nabla}^2\Phi &= \kappa \rho c^2 \\
\frac{1}{c^2}\vec{\nabla}^2\Phi + \frac{1}{2}\left(1 + \frac{2\Phi}{c^2}\right)\frac{2}{c^2}\vec{\nabla}^2\Phi &= \kappa \rho c^2 \\
\frac{1}{c^2}\vec{\nabla}^2\Phi + \frac{1}{2}\frac{2}{c^2}\vec{\nabla}^2\Phi + O(\Phi^2) &= \kappa \rho c^2 \\
\frac{2}{c^2}\vec{\nabla}^2\Phi &= \kappa \rho c^2 \\
\vec{\nabla}^2\Phi &= \frac{1}{2}\kappa \rho c^4 \qquad (117)
\end{aligned}$$

Newtonian gravity is given by the Poisson equation

$$\vec{\nabla}^2\Phi = 4\pi G \rho \qquad (118)$$

and hence $\kappa = \frac{8\pi G}{c^4}$. Finally, the Einstein field equations are

$$\begin{aligned}
G_{\mu\nu} &= \frac{8\pi G}{c^4} T_{\mu\nu} \\
G^{\mu\nu} &= \frac{8\pi G}{c^4} T^{\mu\nu} \qquad (119)
\end{aligned}$$

### 7.3 Summary

What was the path to get to this equation?

- We started with some assumptions: We look for a tensor relation between curvature and matter that conserves energy and momentum, and that recovers Newton's equations.
- The simple-most choice is $G_{\mu\nu} = \kappa T_{\mu\nu}$.
- So we need a tensor for $G$, for which the covariant derivative vanishes - as $\nabla T = 0$.
- Using the Bianchi identity, we were able to construct one, consisting of the Ricci tensor and Ricci scalar. This required some assumptions on the metric: That it is torsion-free and that the metric compatibility holds.
- Using the geodesic equation of motion for a weak field / slow motion, and neglecting higher order terms, we saw that the 00-component of the metric relates to the Newtonian potential.
- With that we looked at the 00-equation of the proposed tensor relation, and were able to derive $\kappa$ such that the Newtonian law is recovered in first order.
- That completes the proposal for the field equations. On whether this was a correct proposal, only experiment can judge.
- The resulting equations are complicated. The Ricci tensor and scalar are functions of the Christoffel symbols and metric tensor, and the Christoffel symbols are functions of the metric themselves - including derivatives. See figure 5.

$$\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\mu g_{\beta\nu} + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\nu g_{\mu\beta} - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} - \frac{3}{2}g^{\alpha\beta}\partial_\mu\partial_\nu g_{\alpha\beta} - \frac{1}{2}g^{\beta\lambda}g^{\alpha\rho}\partial_\alpha g_{\rho\lambda}\partial_\mu g_{\beta\nu}$$
$$- \frac{1}{2}g^{\beta\lambda}g^{\alpha\rho}\partial_\alpha g_{\rho\lambda}\partial_\nu g_{\mu\beta} + \frac{1}{4}g^{\beta\lambda}g^{\alpha\rho}\partial_\nu g_{\alpha\lambda}\partial_\mu g_{\rho\beta} + \frac{1}{4\,|g|}g^{\alpha\beta}\partial_\beta\,|g|\,\partial_\nu g_{\mu\alpha} - \frac{1}{4\,|g|}g^{\alpha\beta}\partial_\beta\,|g|\,\partial_\alpha g_{\mu\nu}$$
$$- \frac{1}{4\,|g|}g^{\alpha\beta}\partial_\beta\,|g|\,\partial_\mu g_{\alpha\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$\frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}g^{\alpha\beta}\partial_\alpha\partial_\mu g_{\beta\nu} + \frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}g^{\alpha\beta}\partial_\alpha\partial_\nu g_{\mu\beta} - \frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} - \frac{3}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}g^{\alpha\beta}\partial_\mu\partial_\nu g_{\alpha\beta} - \frac{1}{2}$$
$$\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\rho=0}^{3}\sum_{\lambda=0}^{3}g^{\beta\lambda}g^{\alpha\rho}\partial_\alpha g_{\rho\lambda}\partial_\mu g_{\beta\nu} - \frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\rho=0}^{3}\sum_{\lambda=0}^{3}g^{\beta\lambda}g^{\alpha\rho}\partial_\alpha g_{\rho\lambda}\partial_\nu g_{\mu\beta} + \frac{1}{4}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\rho=0}^{3}$$
$$\sum_{\lambda=0}^{3}g^{\beta\lambda}g^{\alpha\rho}\partial_\nu g_{\alpha\lambda}\partial_\mu g_{\rho\beta} + \frac{1}{4\,|g|}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}g^{\alpha\beta}\partial_\beta\,|g|\,\partial_\nu g_{\mu\alpha} - \frac{1}{4\,|g|}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}g^{\alpha\beta}\partial_\beta\,|g|\,\partial_\alpha g_{\mu\nu} - \frac{1}{4\,|g|}\sum_{\alpha=0}^{3}$$
$$\sum_{\beta=0}^{3}g^{\alpha\beta}\partial_\beta\,|g|\,\partial_\mu g_{\alpha\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$\frac{1}{2}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}g^{\alpha\beta}\partial_\alpha\partial_\mu g_{\beta\nu} = \frac{1}{2}g^{00}\partial_0\partial_\mu g_{0\nu} + \frac{1}{2}g^{01}\partial_0\partial_\mu g_{1\nu} + \frac{1}{2}g^{02}\partial_0\partial_\mu g_{2\nu} + \frac{1}{2}g^{03}\partial_0\partial_\mu g_{3\nu}$$
$$+ \frac{1}{2}g^{10}\partial_1\partial_\mu g_{0\nu} + \frac{1}{2}g^{11}\partial_1\partial_\mu g_{1\nu} + \frac{1}{2}g^{12}\partial_1\partial_\mu g_{2\nu} + \frac{1}{2}g^{13}\partial_1\partial_\mu g_{3\nu} + \frac{1}{2}g^{20}\partial_2\partial_\mu g_{0\nu} + \frac{1}{2}g^{21}\partial_2\partial_\mu g_{1\nu}$$
$$+ \frac{1}{2}g^{22}\partial_2\partial_\mu g_{2\nu} + \frac{1}{2}g^{23}\partial_2\partial_\mu g_{3\nu} + \frac{1}{2}g^{30}\partial_3\partial_\mu g_{0\nu} + \frac{1}{2}g^{31}\partial_3\partial_\mu g_{1\nu} + \frac{1}{2}g^{32}\partial_3\partial_\mu g_{2\nu} + \frac{1}{2}g^{33}\partial_3\partial_\mu g_{3\nu}$$

Figure 5: Expanding the Einstein equations shows how complicated they are. Top: The full equation in terms of the metric tensor (also including the cosmological constant term). Middle: The same, but the sums explicitly spelled out. Bottom: The first term with the sums expanded. Source: Ville Hirvonen

# 8 Derivation of the Schwarzschild solution

The simple-most solution one can think of is:

- spherically symmetric spacetime: invariant under rotations and taking the mirror image. The symmetry also suggests using spherical coordinates.
- static spacetime: all metric components are independent of the time coordinate $t$ and under time reversal
- vacuum solution
- asymptotically, it should become flat, such that one can embed it into flat Minkowski space

## 8.1 Exploiting symmetries

Exploit time reversal coordinate transformation: $(t, \vec{x}) \longrightarrow (-t, \vec{x})$ . The metric components should stay unchanged. For $i = 1, 2, 3$:

$$g'_{i0} = \frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{\partial x'^0} g_{\alpha\beta} = \frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{c\partial t'} g_{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^i}\frac{\partial x^\beta}{-c\partial t} g_{\alpha\beta} = -\frac{\partial x^\alpha}{\partial x^i}\frac{\partial x^\beta}{\partial x^0} g_{\alpha\beta} = -g_{i0} \tag{120}$$

Thus $g_{i0} = 0 = g_{0i}$. Exploiting spatial symmetries: $(ct, r, \theta, \phi) \longrightarrow (ct, r, -\theta, \phi)$ and $(ct, r, \theta, \phi) \longrightarrow (ct, r, \theta, -\phi)$ yield

$$\begin{aligned} g_{\mu 2} = g_{2\mu} &= 0 \ (\mu \neq 2) \\ g_{\mu 3} = g_{3\mu} &= 0 \ (\mu \neq 3) \end{aligned} \tag{121}$$

Together: $g_{\mu\nu} = 0$ for $\mu \neq \nu$. Thus the metric is diagonal:

$$\begin{aligned} ds^2 &= g_{00}c^2dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 \\ &= g_{tt}c^2dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \end{aligned} \tag{122}$$

Since the metric is static, none of $g_{00}, g_{rr}, g_{\theta\theta}, g_{\phi\phi}$ can depend on $t$. Also, we can exploit the spherical symmetry: On a radial line (a "hypersurface" of constant $t, \theta, \phi$) $g_{rr}$ can only depend on $r$:

$$g_{rr} = A(r) \tag{123}$$

Similarly, $g_{tt}$ can only depend on $r$:

$$g_{tt} = -B(r) \tag{124}$$

On a hypersurface of constant $t_0$ and constant $r_0$, the metric must be that of a sphere:

$$g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 = dl^2 = r_0^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{125}$$

Hence, $g_{\theta\theta} = r_0^2$, $g_{\phi\phi} = r_0^2 \sin^2\theta$. As this holds for any $t_0$, $r_0$, we have

$$\begin{aligned} g_{\theta\theta} &= r^2 \\ g_{\phi\phi} &= r^2 \sin^2\theta \end{aligned} \tag{126}$$

We thus get

$$ds^2 = -B(r)c^2dt^2 + A(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{127}$$

or

$$g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{128}$$

## 8.2 Calculation of the Christoffel symbols

Next, we need to evaluate the $4^3 = 64$ Christoffel symbols $\Gamma^\lambda{}_{\mu\nu}$. Due to the symmetry in $\mu, \nu$, there are actually fewer, namely 40.

### 8.2.1 $\Gamma^0{}_{\mu\nu}$

Let's start with $\Gamma^0{}_{\mu\nu}$:

$$\Gamma^0{}_{00} = \frac{1}{2}g^{0\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}) = -\frac{1}{2}g^{0\alpha}\partial_\alpha g_{00} = -\frac{1}{2}g^{00}\partial_0 g_{00} = 0 \tag{129}$$

(The second equality holds since the metric static, i.e. $\partial_0 g_{\mu\nu}$ vanishes, the third uses $g^{0i} = 0$ and the last again $\partial_0 \longrightarrow 0$.)

$$\Gamma^0{}_{10} = \Gamma^0{}_{01} = \frac{1}{2}g^{0\alpha}(\partial_0 g_{\alpha 1} + \partial_1 g_{\alpha 0} - \partial_\alpha g_{01}) = \frac{1}{2}g^{00}(0 + \partial_1 g_{00} - 0) = \frac{1}{2}(g_{00})^{-1}\partial_r g_{00} = \frac{B'(r)}{2B(r)} \tag{130}$$

$$\begin{aligned}
\Gamma^0{}_{20} = \Gamma^0{}_{02} &= \frac{1}{2}g^{0\alpha}(\partial_0 g_{\alpha 2} + \partial_2 g_{\alpha 0} - \partial_\alpha g_{02}) = \frac{1}{2}g^{00}(0 + \partial_2 g_{00} - 0) = 0 \\
\Gamma^0{}_{30} = \Gamma^0{}_{03} &= \frac{1}{2}g^{0\alpha}(\partial_0 g_{\alpha 3} + \partial_3 g_{\alpha 0} - \partial_\alpha g_{03}) = \frac{1}{2}g^{00}(0 + \partial_3 g_{00} - 0) = 0
\end{aligned} \tag{131}$$

(because $g_{00} = -B(r)$ is not a function of $\theta$ or $\phi$.)

$$\Gamma^0{}_{11} = \frac{1}{2}g^{0\alpha}(\partial_1 g_{\alpha 1} + \partial_1 g_{\alpha 1} - \partial_\alpha g_{11}) = -\frac{1}{2}g^{0\alpha}\partial_\alpha g_{11} = -\frac{1}{2}g^{00}\partial_0 g_{11} = 0 \tag{132}$$

(the second equality is because $g_{\alpha 1}$ is $\neq 0$ only for $\alpha = 1$, but then $g^{0\alpha}$ vanishes; the third is because $g^{0\alpha}$ is $\neq 0$ only for $\alpha = 0$, and the last is using $\partial_0 \longrightarrow 0$)

$$\begin{aligned}
\Gamma^0{}_{12} = \Gamma^0{}_{21} &= \frac{1}{2}g^{0\alpha}(\partial_1 g_{\alpha 2} + \partial_2 g_{\alpha 1} - \partial_\alpha g_{12}) = (0 + 0 - 0) = 0 \\
\Gamma^0{}_{13} = \Gamma^0{}_{31} &= \frac{1}{2}g^{0\alpha}(\partial_1 g_{\alpha 3} + \partial_3 g_{\alpha 1} - \partial_\alpha g_{13}) = (0 + 0 - 0) = 0 \\
\Gamma^0{}_{23} = \Gamma^0{}_{32} &= \frac{1}{2}g^{0\alpha}(\partial_2 g_{\alpha 3} + \partial_3 g_{\alpha 2} - \partial_\alpha g_{23}) = (0 + 0 - 0) = 0
\end{aligned} \tag{133}$$

(Line 1: The first term vanishes as $g_{\alpha 2}$ is $\neq 0$ only for $\alpha = 2$, but then $g^{0\alpha} = 0$. The second as $g_{\alpha 1} \neq 0$ only for $\alpha = 1$, but then $g^{0\alpha} = 0$. The last as $g_{12} = 0$. Line 2: The first term vanishes as $g_{\alpha 3}$ is $\neq 0$ only for $\alpha = 3$, but then $g^{0\alpha} = 0$. The second as $g_{\alpha 1} \neq 0$ only for $\alpha = 1$, but then $g^{0\alpha} = 0$. The last as $g_{13} = 0$. Line 3: The first term vanishes as $g_{\alpha 3}$ is $\neq 0$ only for $\alpha = 3$, but then $g^{0\alpha} = 0$. The second term vanishes as $g_{\alpha 2}$ is $\neq 0$ only for $\alpha = 3$, but then $g^{0\alpha} = 0$. The last as $g_{23} = 0$.)

$$\begin{aligned}
\Gamma^0{}_{22} &= \frac{1}{2}g^{0\alpha}(\partial_2 g_{\alpha 2} + \partial_2 g_{\alpha 2} - \partial_\alpha g_{22}) = -\frac{1}{2}g^{0\alpha}\partial_\alpha g_{22} = -\frac{1}{2}g^{00}\partial_0 g_{22} = 0 \\
\Gamma^0{}_{33} &= \frac{1}{2}g^{0\alpha}(\partial_3 g_{\alpha 3} + \partial_3 g_{\alpha 3} - \partial_\alpha g_{33}) = -\frac{1}{2}g^{0\alpha}\partial_\alpha g_{33} = -\frac{1}{2}g^{00}\partial_0 g_{33} = 0
\end{aligned} \tag{134}$$

(The second equality is because $g^{0\alpha} \neq 0$ only for $\alpha = 0$, but then $g_{\alpha 2/3} = 0$, the third as $g^{0\alpha} \neq 0$ only for $\alpha = 0$, the last as $\partial_0 \longrightarrow 0$.) Thus:

$$\Gamma^0{}_{\mu\nu} = \begin{pmatrix} 0 & \frac{B'(r)}{2B(r)} & 0 & 0 \\ \frac{B'(r)}{2B(r)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{135}$$

#### 8.2.2 $\Gamma^1{}_{\mu\nu}$

Next, let's do $\Gamma^1{}_{\mu\nu}$.

$$\Gamma^1{}_{00} = \frac{1}{2}g^{1\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}) = -\frac{1}{2}g^{1\alpha}\partial_\alpha g_{00} = -\frac{1}{2}g^{11}\partial_1 g_{00} = -\frac{1}{2}(g_{11})^{-1}\partial_r g_{00} = \frac{B'(r)}{2A(r)} \tag{136}$$

For $i = 1, 2, 3$:

$$\Gamma^1{}_{0i} = \Gamma^1{}_{i0} = \frac{1}{2}g^{1\alpha}(\partial_0 g_{\alpha i} + \partial_i g_{\alpha 0} - \partial_\alpha g_{0i}) = \frac{1}{2}g^{11}(\partial_i g_{10} - \partial_1 g_{0i}) = 0 - 0 = 0 \tag{137}$$

(The second equality uses that only $\alpha = 1$ can contribute and that $\partial_0 \longrightarrow 0$. The third holds as $g_{10} = 0$ and $g_{0i} = 0$.)

$$\begin{aligned}\Gamma^1{}_{11} &= \frac{1}{2}g^{1\alpha}(\partial_1 g_{\alpha 1} + \partial_1 g_{\alpha 1} - \partial_\alpha g_{11}) = \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) \\ &= \frac{1}{2}g^{11}\partial_1 g_{11} = \frac{1}{2}(g_{11})^{-1}\partial_r g_{11} = \frac{A'(r)}{2A(r)}\end{aligned} \tag{138}$$

$$\begin{aligned}\Gamma^1{}_{12} = \Gamma^1{}_{21} &= \frac{1}{2}g^{1\alpha}(\partial_1 g_{\alpha 2} + \partial_2 g_{\alpha 1} - \partial_\alpha g_{12}) = \frac{1}{2}g^{11}(\partial_1 g_{12} + \partial_2 g_{11} - 0) = 0 + 0 = 0 \\ \Gamma^1{}_{13} = \Gamma^1{}_{31} &= \frac{1}{2}g^{1\alpha}(\partial_1 g_{\alpha 3} + \partial_3 g_{\alpha 1} - \partial_\alpha g_{13}) = \frac{1}{2}g^{11}(\partial_1 g_{13} + \partial_3 g_{11} - 0) = 0 + 0 = 0 \\ \Gamma^1{}_{23} = \Gamma^1{}_{32} &= \frac{1}{2}g^{1\alpha}(\partial_2 g_{\alpha 3} + \partial_3 g_{\alpha 2} - \partial_\alpha g_{23}) = \frac{1}{2}g^{11}(\partial_2 g_{13} + \partial_3 g_{11} - 0) = 0 + 0 = 0\end{aligned} \tag{139}$$

(First line: The second equality uses that only $\alpha = 1$ can contribute and that $g_{12} = 0$. The third uses $g_{12} = 0$ and that $g_{11} = A(r)$ is not a function of $\theta$. Second line: The second equality uses that only $\alpha = 1$ can contribute and that $g_{13} = 0$. The third uses $g_{13} = 0$ and that $g_{11} = A(r)$ is not a function of $\phi$. Third line: The second equality uses that only $\alpha = 1$ can contribute and that $g_{23} = 0$. The third uses $g_{13} = 0$ and that $g_{11} = A(r)$ is not a function of $\phi$.)

$$\begin{aligned}\Gamma^1{}_{22} &= \frac{1}{2}g^{1\alpha}(\partial_2 g_{\alpha 2} + \partial_2 g_{\alpha 2} - \partial_\alpha g_{22}) = \frac{1}{2}g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) \\ &= -\frac{1}{2}g^{11}\partial_r g_{22} = -\frac{1}{2}(g_{11})^{-1}\partial_r g_{22} = -\frac{1}{2A(r)}\partial_r r^2 = -\frac{r}{A(r)}\end{aligned} \tag{140}$$

$$\begin{aligned}\Gamma^1{}_{33} &= \frac{1}{2}g^{1\alpha}(\partial_3 g_{\alpha 3} + \partial_3 g_{\alpha 3} - \partial_\alpha g_{33}) = \frac{1}{2}g^{11}(\partial_3 g_{12} + \partial_3 g_{12} - \partial_1 g_{33}) \\ &= -\frac{1}{2}g^{11}\partial_r g_{33} = -\frac{1}{2}(g_{11})^{-1}\partial_r g_{33} = -\frac{1}{2A(r)}\partial_r r^2 \sin^2\theta = -\frac{r\sin^2\theta}{A(r)}\end{aligned} \tag{141}$$

Thus:

$$\Gamma^1{}_{\mu\nu} = \begin{pmatrix} \frac{B'(r)}{2A(r)} & 0 & 0 & 0 \\ 0 & \frac{A'(r)}{2A(r)} & 0 & 0 \\ 0 & 0 & -\frac{r}{A(r)} & 0 \\ 0 & 0 & 0 & -\frac{r\sin^2\theta}{A(r)} \end{pmatrix} \tag{142}$$

### 8.2.3 $\Gamma^2{}_{\mu\nu}$

Moving on to the $\Gamma^2{}_{\mu\nu}$.

$$\Gamma^2{}_{00} = \frac{1}{2}g^{2\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}) = -\frac{1}{2}g^{22}\partial_2 g_{00} = 0 \tag{143}$$

(Using $\partial_0 \longrightarrow 0$ and that $g_{00} = -B(r)$ is not a function of $\theta$.) For $i = 1, 2, 3$:

$$\Gamma^2{}_{0i} = \Gamma^2{}_{i0} = \frac{1}{2}g^{2\alpha}(\partial_0 g_{\alpha i} + \partial_i g_{\alpha 0} - \partial_\alpha g_{0i}) = \frac{1}{2}g^{22}(\partial_i g_{20} - \partial_1 g_{02}) = 0 - 0 = 0 \tag{144}$$

(Using $\partial_0 \longrightarrow 0$, that only $\alpha = 2$ can contribute and that $g_{20} = g_{02} = 0$).

$$\Gamma^2{}_{11} = \frac{1}{2}g^{2\alpha}(\partial_1 g_{\alpha 1} + \partial_1 g_{\alpha 1} - \partial_\alpha g_{11}) = \frac{1}{2}g^{22}(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}) = 0 + 0 - 0 = 0 \tag{145}$$

(Using that only $\alpha = 2$ can contribute and that $g_{11} = A(r)$ is not a function of $\theta$.)

$$\begin{aligned}\Gamma^2{}_{12} = \Gamma^2{}_{21} &= \frac{1}{2}g^{2\alpha}(\partial_1 g_{\alpha 2} + \partial_2 g_{\alpha 1} - \partial_\alpha g_{12}) = \frac{1}{2}g^{22}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{21}) \\ &= \frac{1}{2}(g_{22})^{-1}\partial_1 g_{22} = \frac{1}{2}\frac{1}{r^2}\partial_r r^2 = \frac{1}{r}\end{aligned} \tag{146}$$

$$\Gamma^2{}_{22} = \frac{1}{2}g^{2\alpha}(\partial_2 g_{\alpha 2} + \partial_2 g_{\alpha 2} - \partial_\alpha g_{22}) = \frac{1}{2}g^{22}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) = 0 + 0 - 0 = 0 \tag{147}$$

($g_{22} = r^2$ does not depend $\theta$).

$$\begin{aligned}\Gamma^2{}_{33} &= \frac{1}{2}g^{2\alpha}(\partial_3 g_{\alpha 3} + \partial_3 g_{\alpha 3} - \partial_\alpha g_{33}) = \frac{1}{2}g^{22}(\partial_3 g_{23} + \partial_2 g_{23} - \partial_2 g_{33}) \\ &= -\frac{1}{2}(g_{22})^{-1}\partial_\theta g_{33} = -\frac{1}{2r^2}\partial_\theta r^2\cos^2\theta = -\cos\theta\,\sin\theta\end{aligned} \tag{148}$$

$$\Gamma^2{}_{13} = \Gamma^2{}_{31} = \frac{1}{2}g^{2\alpha}(\partial_1 g_{\alpha 3} + \partial_3 g_{\alpha 1} - \partial_\alpha g_{13}) = \frac{1}{2}g^{22}(\partial_1 g_{23} + \partial_3 g_{21} - 0) = 0 \tag{149}$$

(only $\alpha = 2$ contributes)

$$\Gamma^2{}_{23} = \Gamma^2{}_{32} = \frac{1}{2}g^{2\alpha}(\partial_2 g_{\alpha 3} + \partial_3 g_{\alpha 2} - \partial_\alpha g_{23}) = \frac{1}{2}g^{22}(\partial_2 g_{23} + \partial_3 g_{22} - 0) = 0 \tag{150}$$

(only $\alpha = 2$ contributes, and $g_{22} = r^2$ is not a function $\phi$.) Thus:

$$\Gamma^2{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\,\sin\theta \end{pmatrix} \tag{151}$$

### 8.2.4 $\Gamma^3{}_{\mu\nu}$

And finally the $\Gamma^3{}_{\mu\nu}$.

$$\Gamma^3{}_{00} = \frac{1}{2}g^{3\alpha}(\partial_0 g_{\alpha 0} + \partial_0 g_{\alpha 0} - \partial_\alpha g_{00}) = -\frac{1}{2}g^{33}\partial_3 g_{00} = 0 \tag{152}$$

(Using $\partial_0 \longrightarrow 0$ and that $g_{00} = -B(r)$ is not a function of $\phi$.) For $i = 1, 2, 3$:

$$\Gamma^3{}_{0i} = \Gamma^3{}_{i0} = \frac{1}{2}g^{3\alpha}(\partial_0 g_{\alpha i} + \partial_i g_{\alpha 0} - \partial_\alpha g_{0i}) = \frac{1}{2}g^{33}(\partial_i g_{30} - \partial_3 g_{03}) = 0 - 0 = 0 \tag{153}$$

$$\Gamma^3{}_{11} = \frac{1}{2}g^{3\alpha}(\partial_1 g_{\alpha 1} + \partial_1 g_{\alpha 1} - \partial_\alpha g_{11}) = \frac{1}{2}g^{33}(\partial_1 g_{31} + \partial_1 g_{31} - \partial_3 g_{11}) = 0 + 0 - 0 = 0 \tag{154}$$

(Using that only $\alpha = 3$ can contribute, that $g_{31} = 0$ and that $g_{11} = A(r)$ is not a function of $\phi$.)

$$\Gamma^3{}_{12} = \Gamma^3{}_{21} = \frac{1}{2}g^{3\alpha}(\partial_1 g_{\alpha 2} + \partial_2 g_{\alpha 1} - \partial_\alpha g_{12}) = \frac{1}{2}g^{33}(\partial_1 g_{32} + \partial_2 g_{31} - \partial_3 g_{21}) = 0 \tag{155}$$

$$\Gamma^3{}_{22} = \frac{1}{2}g^{3\alpha}(\partial_2 g_{\alpha 2} + \partial_2 g_{\alpha 2} - \partial_\alpha g_{22}) = \frac{1}{2}g^{33}(\partial_2 g_{32} + \partial_2 g_{32} - \partial_3 g_{22}) = 0 \tag{156}$$

($g_{22} = r^2$ is not a function $\phi$.)

$$\begin{aligned} \Gamma^3{}_{13} = \Gamma^3{}_{31} &= \frac{1}{2}g^{3\alpha}(\partial_1 g_{\alpha 3} + \partial_3 g_{\alpha 1} - \partial_\alpha g_{13}) = \frac{1}{2}g^{33}(\partial_1 g_{33} + \partial_3 g_{31} - 0) \\ &= \frac{1}{2}(g_{33})^{-1}\partial_r\, r^2 \sin^2\theta = \frac{1}{2}\frac{1}{r^2\sin^2\theta}2r\sin^2\theta = \frac{1}{r} \end{aligned} \tag{157}$$

$$\begin{aligned} \Gamma^3{}_{23} = \Gamma^3{}_{32} &= \frac{1}{2}g^{3\alpha}(\partial_2 g_{\alpha 3} + \partial_3 g_{\alpha 2} - \partial_\alpha g_{23}) = \frac{1}{2}g^{33}(\partial_2 g_{33} + \partial_3 g_{32} - 0) \\ &= \frac{1}{2}(g_{33})^{-1}\partial_\theta\, r^2 \sin^2\theta = \frac{1}{2}\frac{1}{r^2\sin^2\theta}r^2 2\cos\theta\sin\theta = \cot\theta \end{aligned} \tag{158}$$

Thus:

$$\Gamma^3{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix} \tag{159}$$

We are done!

### 8.2.5 Summary of Christoffel symbols calculated

For the following, it is handy to have all non-vanishing Christoffel symbols collected.

$$\begin{array}{llll} \Gamma^0{}_{10} = \Gamma^0{}_{01} = \frac{B'}{2B} & \Gamma^1{}_{00} = \frac{B'}{2A} & \Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r} & \Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r} \\ & \Gamma^1{}_{11} = \frac{A'}{2A} & \Gamma^2{}_{33} = -\cos\theta\sin\theta & \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta \\ & \Gamma^1{}_{22} = -\frac{r}{A} & & \\ & \Gamma^1{}_{33} = -\frac{r\sin^2\theta}{A} & & \end{array} \tag{160}$$

### 8.2.6 Tricks for Christoffels

For contracted Christoffel symbols $\Gamma^{\mu}{}_{\mu\nu} = \Gamma^{\mu}{}_{\nu\mu}$ (these occur in two of the four terms for the Ricci tensore) exists a neat trick.

$$\begin{aligned}
\Gamma^{\mu}{}_{\mu\nu} &= \frac{1}{2} g^{\mu\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}) \\
&= \frac{1}{2} g^{\mu\alpha}\partial_\mu g_{\alpha\nu} - \frac{1}{2} g^{\mu\alpha}\partial_\alpha g_{\mu\nu} + \frac{1}{2} g^{\mu\alpha}\partial_\nu g_{\mu\alpha} \\
&= \frac{1}{2} g^{\mu\alpha}\partial_\mu g_{\alpha\nu} - \frac{1}{2} g^{\alpha\mu}\partial_\mu g_{\alpha\nu} + \frac{1}{2} g^{\mu\alpha}\partial_\nu g_{\mu\alpha} \\
&= \frac{1}{2} g^{\mu\alpha}\partial_\nu g_{\mu\alpha} \qquad (161)
\end{aligned}$$

In line 3 we changed the names of the summation indices $\mu$ to $\alpha$ and $\alpha$ to $\mu$. The trick is based on the linear algebra fact that for any matrix

$$\ln |M| = \mathrm{tr}(\ln M) \qquad (162)$$

where $|M|$ is the determinant of the matrix. Take the derivative $\partial_\nu$ on the left side:

$$\partial_\nu \ln |M| = \frac{1}{|M|}\partial_\nu |M| \qquad (163)$$

And on the right side:

$$\partial_\nu \mathrm{tr}(\ln M) = \mathrm{tr}(\partial_\nu \ln M) = \mathrm{tr}(M^{-1}\partial_\nu M) = M^{\mu\alpha}\partial_\nu M_{\mu\alpha} \qquad (164)$$

This is just what we found in equation 161 for the Christoffel symbol, if $M = g$. Thus

$$\Gamma^{\mu}{}_{\mu\nu} = \Gamma^{\mu}{}_{\nu\mu} = \frac{1}{2|g|}\partial_\nu |g| = \partial_\nu \ln \sqrt{|g|} \qquad (165)$$

The formula is particularly useful for diagonal metrics, since $|g| = g_{00}g_{11}g_{22}g_{33}$ is just the product of the diagonal elements. For example, for our ansatz for the Schwarzschild metric, $|g| = A\, B\, r^4 \sin^2\theta$ yields directly

$$\Gamma^{0}{}_{01} + \Gamma^{1}{}_{11} + \Gamma^{2}{}_{21} + \Gamma^{3}{}_{31} = \frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r} \qquad (166)$$

Another useful way to get the Christoffel symbols is exploiting section 5. From the Lagrangian $\mathcal{L}'$ (which is essentially 1/2 times the line element divided by $d\tau^2$!), one gets the equation of motions via the Euler-Lagrange mechanism. These are four equations, for each of the coordinates one for its second derivative. The geodesic equation is identical to these, and in its definition one has the Christoffel symbols. Hence, one writes the four equations for $\ddot{x}^0, \ddot{x}^1, \ddot{x}^2, \ddot{x}^3$, each of which will contain four terms for the squares of velocities and six cross terms. By comparing the coefficients, one can read off the Christoffel symbols. Let's try this. The geodesic equation written out for component $\mu$ is

$$\begin{aligned}
-\ddot{x}^\mu &= \Gamma^{\mu}{}_{00}(u^0)^2 + 2\Gamma^{\mu}{}_{01}u^0u^1 + 2\Gamma^{\mu}{}_{02}u^0u^2 + 2\Gamma^{\mu}{}_{03}u^0u^3 + \\
&\quad \Gamma^{\mu}{}_{11}(u^1)^2 + 2\Gamma^{\mu}{}_{12}u^1u^2 + 2\Gamma^{\mu}{}_{13}u^1u^3 + \\
&\quad \Gamma^{\mu}{}_{22}(u^2)^2 + 2\Gamma^{\mu}{}_{23}2^0u^3 + \\
&\quad \Gamma^{\mu}{}_{33}(u^3)^2 \qquad (167)
\end{aligned}$$

Let's look at our ansatz for the Schwarzschild metric:

$$\mathcal{L}' = -\frac{1}{2}B(r)(u^t)^2 + \frac{1}{2}A(r)(u^r)^2 + \frac{1}{2}r^2(u^\theta)^2 + \frac{1}{2}r^2\sin^2\theta(u^\phi)^2 \qquad (168)$$

For the 0-component, we have

$$\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial x^t} &= 0 \\
\frac{d}{d\tau}\frac{\partial \mathcal{L}'}{\partial u^t} &= -\frac{d}{d\tau}(B(r)u^t) = -B'\dot{r}u^t - B\dot{u}^t = -B'u^r u^t - B\ddot{x}^t \\
-\ddot{x}^0 = -\ddot{x}^t &= \frac{B'}{B}u^t u^r = \frac{B'}{B}u^0 u^1
\end{aligned} \tag{169}$$

Comparing the coefficients with equation 167, one sees that only ${\Gamma^0}_{10} = {\Gamma^0}_{01}$ does not vanish, and we find (like before) ${\Gamma^0}_{10} = {\Gamma^0}_{01} = B'/(2B)$ as in equation 160 The 1-component is slightly more work:

$$\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial r} &= -\frac{1}{2}B'(u^t)^2 + \frac{1}{2}A'(u^r)^2 + r(u^\theta)^2 + r\sin^2\theta(u^\phi)^2 \\
\frac{d}{d\tau}\frac{\partial \mathcal{L}'}{\partial u^r} &= \frac{d}{d\tau}(A(r)u^r) = A'\dot{r}u^r + A\dot{u}^r = A'(u^r)^2 + A\ddot{x}^r \\
-\ddot{x}^1 = -\ddot{x}^r &= \frac{A'}{A}(u^r)^2 + \frac{1}{2}\frac{B'}{A}(u^t)^2 - \frac{1}{2}\frac{A'}{A}(u^r)^2 - \frac{r}{A}(u^\theta)^2 - \frac{r}{A}\sin^2\theta(u^\phi)^2 \\
&= \frac{1}{2}\frac{A'}{A}(u^r)^2 + \frac{1}{2}\frac{B'}{A}(u^t)^2 - \frac{r}{A}(u^\theta)^2 - \frac{r}{A}\sin^2\theta(u^\phi)^2
\end{aligned} \tag{170}$$

Again, we can confirm by comparing coefficients with equation 167 that equation 160 is correct, now for the 1-component. The 2-component:

$$\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial \theta} &= r^2\sin\theta\cos\theta(u^\phi)^2 \\
\frac{d}{d\tau}\frac{\partial \mathcal{L}'}{\partial u^t} &= \frac{d}{d\tau}(r^2u^\theta) = 2r\,\dot{r}u^\theta + r^2\dot{u}^\theta = 2ru^r u^\theta + r^2\ddot{x}^\theta \\
-\ddot{x}^2 = -\ddot{x}^\theta &= \frac{2}{r}u^r u^\theta - \sin\theta\cos\theta(u^\phi)^2 = \frac{2}{r}u^1u^2 - \sin\theta\cos\theta(u^3)^2
\end{aligned} \tag{171}$$

Also here, we get by comparing coefficients with equation 167 the correct Christoffel symbols from equation 160. Finally the 3-component:

$$\begin{aligned}
\frac{\partial \mathcal{L}'}{\partial \phi} &= 0 \\
\frac{d}{d\tau}\frac{\partial \mathcal{L}'}{\partial u^\phi} &= \frac{d}{d\tau}(r^2\sin^2\theta u^\phi) = 2r\,u^r\sin^2\theta u^\phi + 2r^2\sin\theta\cos\theta\,u^\theta u^\phi + r^2\sin^2\theta\,\ddot{x}^\phi \\
-\ddot{x}^3 = -\ddot{x}^\phi &= \frac{2}{r}u^r u^\phi + \cot\theta\,u^\theta u^\phi = \frac{2}{r}u^1u^3 + \cot\theta\,u^2u^3
\end{aligned} \tag{172}$$

And also the final comparison with equation 167 confirms what we had in equation 160. Overall, this way of calculating the Christoffel symbols seems more elegant, faster and less error-prone.

## 8.3 The vacuum solution

Vacuum means $T_{\mu\nu} = 0$, i.e. we have $G_{\mu\nu} = 0$, which implies $R_{\mu\nu} = 0$ (as then the Ricci scalar also vanishes). We have thus to solve the $4^2 = 16$ field equations. It works simply by brute-force: Plugging into the Ricci tensor the Christoffel symbols just retrieved. What one uses then repeatedly is:

- Many Christoffel symbols are 0.
- Time derivatives $\partial_0$ are 0 as the metric is static.
- The coordinates are independent, i.e. a function f only depending on $\theta$ does not depend on $r$: $\partial_r f(\theta) = 0$
- The product rule of derivation: $(fg)' = f'g + fg'$
- Expanding out the contracted sums, like $\Gamma^\lambda{}_{\mu\lambda} = \Gamma^0{}_{\mu 0} + \Gamma^1{}_{\mu 1} + \Gamma^2{}_{\mu 2} + \Gamma^3{}_{\mu 3}$

All this is not difficult, just cumbersome.

$$\begin{aligned}
R_{00} &= R^\lambda{}_{0\lambda 0} = \partial_\beta \Gamma^\beta{}_{00} - \partial_0 \Gamma^\beta{}_{0\beta} + \Gamma^\beta{}_{00}\Gamma^\sigma{}_{\beta\sigma} - \Gamma^\beta{}_{\sigma 0}\Gamma^\sigma{}_{0\beta} \\
&= \partial_0 \Gamma^0{}_{00} + \partial_1 \Gamma^1{}_{00} + \partial_2 \Gamma^2{}_{00} + \partial_3 \Gamma^3{}_{00} - 0 + ... \\
&= 0 + \partial_r \frac{B'(r)}{2A(r)} + 0 + 0 + \Gamma^0{}_{00}\Gamma^\sigma{}_{0\sigma} + \Gamma^1{}_{00}\Gamma^\sigma{}_{1\sigma} + \Gamma^2{}_{00}\Gamma^\sigma{}_{2\sigma} + \Gamma^3{}_{00}\Gamma^\sigma{}_{3\sigma} - ... \\
&= \frac{B''(r)}{2A(r)} - \frac{B'(r)}{2A^2(r)}A'(r) + 0 + \frac{B'(r)}{2A(r)}(\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12} + \Gamma^3{}_{13}) + 0 + 0 - ... \\
&= \frac{B''}{2A} - \frac{B'}{2A^2}A' + \frac{B'}{2A}(\frac{B'}{2B} + \frac{A'}{2A} + \frac{1}{r} + \frac{1}{r}) - ... \\
&= \text{---} - (\Gamma^0{}_{00}\Gamma^0{}_{00} + \Gamma^0{}_{10}\Gamma^1{}_{00} + \Gamma^0{}_{20}\Gamma^2{}_{00} + \Gamma^0{}_{30}\Gamma^3{}_{00} \\
&\quad + \Gamma^1{}_{00}\Gamma^0{}_{01} + \Gamma^1{}_{10}\Gamma^1{}_{01} + \Gamma^1{}_{20}\Gamma^2{}_{01} + \Gamma^1{}_{30}\Gamma^3{}_{01} \\
&\quad + \Gamma^2{}_{00}\Gamma^0{}_{02} + \Gamma^2{}_{10}\Gamma^1{}_{02} + \Gamma^2{}_{20}\Gamma^2{}_{02} + \Gamma^2{}_{30}\Gamma^3{}_{02} \\
&\quad + \Gamma^3{}_{00}\Gamma^0{}_{03} + \Gamma^3{}_{10}\Gamma^1{}_{03} + \Gamma^3{}_{20}\Gamma^2{}_{03} + \Gamma^3{}_{30}\Gamma^3{}_{03}) \\
&= \text{---} - [(0 + \frac{B'}{2B}\frac{B'}{2A} + 0 + 0) + (\frac{B'}{2A}\frac{B'}{2B} + 0 + 0 + 0) + (4 \times 0) + (4 \times 0)] \\
&= \frac{B''}{2A} - \frac{A'B'}{2A^2} + \frac{B'^2}{4AB} + \frac{A'B'}{4A^2} + \frac{B'}{rA} - \frac{B'^2}{2BA} \\
&= \frac{B''}{2A} - \frac{A'B'}{4A^2} - \frac{B'^2}{4AB} + \frac{B'}{rA}
\end{aligned} \tag{173}$$

With $R_{00} = 0$ we thus get

$$0 = 2rABB'' - rBA'B' - rAB'^2 + 4ABB' \tag{174}$$

$$
\begin{aligned}
R_{11} &= R^{\lambda}{}_{1\lambda 1} = \partial_\beta \Gamma^{\beta}{}_{11} - \partial_1 \Gamma^{\beta}{}_{1\beta} + \Gamma^{\beta}{}_{11}\Gamma^{\sigma}{}_{\beta\sigma} - \Gamma^{\beta}{}_{\sigma 1}\Gamma^{\sigma}{}_{1\beta} \\
&= \partial_0 \Gamma^0{}_{11} + \partial_1 \Gamma^1{}_{11} + \partial_2 \Gamma^2{}_{11} + \partial_3 \Gamma^3{}_{11} - \partial_1(\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12} + \Gamma^3{}_{13}) + ... \\
&= 0 + \partial_r \frac{A'(r)}{2A(r)} + 0 + 0 - \partial_r\left(\frac{B'(r)}{2B(r)} + \frac{A'(r)}{2A(r)} + \frac{1}{r} + \frac{1}{r}\right) + ... \\
&= -\frac{B''}{2B} + \frac{B'}{2B^2}B' + \frac{2}{r^2} + \Gamma^0{}_{11}\Gamma^{\sigma}{}_{0\sigma} + \Gamma^1{}_{11}\Gamma^{\sigma}{}_{1\sigma} + \Gamma^2{}_{11}\Gamma^{\sigma}{}_{2\sigma} + \Gamma^3{}_{11}\Gamma^{\sigma}{}_{3\sigma} - ... \\
&= -\frac{B''}{2B} + \frac{B'^2}{2B^2} + \frac{2}{r^2} + 0 + \frac{A'}{2A}(\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12} + \Gamma^3{}_{13}) + 0 + 0 - ... \\
&= -\frac{B''}{2B} + \frac{B'^2}{2B^2} + \frac{2}{r^2} + \frac{A'}{2A}\left(\frac{B'}{2B} + \frac{A'}{2A} + \frac{1}{r} + \frac{1}{r}\right) - ... \\
&= -\frac{B''}{2B} + \frac{B'^2}{2B^2} + \frac{2}{r^2} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{A'}{rA} - ... \\
&= \text{---} - (\Gamma^0{}_{01}\Gamma^0{}_{10} + \Gamma^0{}_{11}\Gamma^1{}_{10} + \Gamma^0{}_{21}\Gamma^2{}_{10} + \Gamma^0{}_{31}\Gamma^3{}_{10} \\
&\quad + \Gamma^1{}_{01}\Gamma^0{}_{11} + \Gamma^1{}_{11}\Gamma^1{}_{11} + \Gamma^1{}_{21}\Gamma^2{}_{11} + \Gamma^1{}_{31}\Gamma^3{}_{11} \\
&\quad + \Gamma^2{}_{01}\Gamma^0{}_{12} + \Gamma^2{}_{11}\Gamma^1{}_{12} + \Gamma^2{}_{21}\Gamma^2{}_{12} + \Gamma^2{}_{31}\Gamma^3{}_{12} \\
&\quad + \Gamma^3{}_{01}\Gamma^0{}_{13} + \Gamma^3{}_{11}\Gamma^1{}_{13} + \Gamma^3{}_{21}\Gamma^2{}_{13} + \Gamma^3{}_{31}\Gamma^3{}_{13}) \\
&= \text{---} - \left[\left(\frac{B'^2}{4B^2} + 0 + 0 + 0\right) + \left(0 + \frac{A'^2}{4A^2} + 0 + 0\right) + \left(0 + 0 + \frac{1}{r^2} + 0\right) + \left(0 + 0 + 0 + \frac{1}{r^2}\right)\right] \\
&= -\frac{B''}{2B} + \frac{B'^2}{4B^2} + \frac{2}{r^2} + \frac{A'B'}{4AB} + \frac{A'^2}{4A^2} + \frac{A'}{rA} - \frac{A'^2}{4A^2} - \frac{2}{r^2} \\
&= -\frac{B''}{2B} + \frac{B'^2}{4B^2} + \frac{A'B'}{4AB} + \frac{A'}{rA}
\end{aligned} \tag{175}
$$

With $R_{11} = 0$ we thus get

$$0 = -2rABB'' + rAB'^2 + rBA'B' + 4B^2A' \tag{176}$$

Adding the two equations from $R_{00}$ and $R_{11}$ one gets

$$0 = 4ABB' + 4B^2A' = 4B(AB' + BA') = 4B(AB)' \tag{177}$$

If the derivative of the product $AB$ is 0, the product needs to be a constant,

$$A(r)B(r) = K_1 \tag{178}$$

Knowing $A(r)$, we also know $B(r)$ up to the constant factor $K_1$.

So, let's turn to $R_{22}$ :

$$
\begin{aligned}
R_{22} &= R^{\lambda}{}_{2\lambda 2} = \partial_\beta \Gamma^\beta{}_{22} - \partial_2 \Gamma^\beta{}_{2\beta} + \Gamma^\beta{}_{22}\Gamma^\sigma{}_{\beta\sigma} - \Gamma^\beta{}_{\sigma 2}\Gamma^\sigma{}_{2\beta} \\
&= \partial_0 \Gamma^0{}_{22} + \partial_1 \Gamma^1{}_{22} + \partial_2 \Gamma^2{}_{22} + \partial_3 \Gamma^3{}_{22} - \partial_2(\Gamma^0{}_{20} + \Gamma^1{}_{21} + \Gamma^2{}_{22} + \Gamma^3{}_{23}) + ... \\
&= (0 - \partial_r \frac{r}{A(r)} + 0 + 0) - \partial_\theta(0 - \frac{r}{A(r)} + 0 + \cot\theta) + ... \\
&= -\frac{1}{A(r)} + \frac{r}{A^2(r)}A'(r) + \frac{1}{\sin^2\theta} + ... \\
&= -\frac{1}{A} + \frac{rA'}{A^2} + \frac{1}{\sin^2\theta} + \Gamma^0{}_{22}\Gamma^\sigma{}_{0\sigma} + \Gamma^1{}_{22}\Gamma^\sigma{}_{1\sigma} + \Gamma^2{}_{22}\Gamma^\sigma{}_{2\sigma} + \Gamma^3{}_{22}\Gamma^\sigma{}_{3\sigma} - ... \\
&= -\frac{1}{A} + \frac{rA'}{A^2} + \frac{1}{\sin^2\theta} + 0 - \frac{r}{A}(\Gamma^0{}_{10} + \Gamma^1{}_{11} + \Gamma^2{}_{12} + \Gamma^3{}_{13}) + 0 + 0 - ... \\
&= -\frac{1}{A} + \frac{rA'}{A^2} + \frac{1}{\sin^2\theta} - \frac{r}{A}(\frac{B'}{2B} + \frac{A'}{2A} + \frac{1}{r} + \frac{1}{r}) + 0 + 0 - ... \\
&= -\frac{3}{A} + \frac{rA'}{2A^2} + \frac{1}{\sin^2\theta} - \frac{rB'}{2AB} - ... \\
&= \text{---} - (\Gamma^0{}_{02}\Gamma^0{}_{20} + \Gamma^0{}_{12}\Gamma^1{}_{20} + \Gamma^0{}_{22}\Gamma^2{}_{20} + \Gamma^0{}_{32}\Gamma^3{}_{20} \\
&\quad + \Gamma^1{}_{02}\Gamma^0{}_{21} + \Gamma^1{}_{12}\Gamma^1{}_{21} + \Gamma^1{}_{22}\Gamma^2{}_{21} + \Gamma^1{}_{32}\Gamma^3{}_{21} \\
&\quad + \Gamma^2{}_{02}\Gamma^0{}_{22} + \Gamma^2{}_{12}\Gamma^1{}_{22} + \Gamma^2{}_{22}\Gamma^2{}_{22} + \Gamma^2{}_{32}\Gamma^3{}_{22} \\
&\quad + \Gamma^3{}_{02}\Gamma^0{}_{23} + \Gamma^3{}_{12}\Gamma^1{}_{23} + \Gamma^3{}_{22}\Gamma^2{}_{23} + \Gamma^3{}_{32}\Gamma^3{}_{23}) \\
&= \text{---} - [4 \times 0 + (0 + 0 - \frac{1}{A} + 0) + (0 - \frac{1}{A} + 0 + 0) + (0 + 0 + 0 + \cot^2\theta)] \\
&= -\frac{1}{A} + \frac{rA'}{2A^2} + \frac{1}{\sin^2\theta} - \frac{rB'}{2AB} - \cot^2\theta \\
&= -\frac{1}{A} + \frac{rA'}{2A^2} + 1 - \frac{rB'}{2AB} \qquad (179)
\end{aligned}
$$

With $R_{\theta\theta} = 0$ we thus get

$$0 \quad = \quad -2AB + rBA' + 2A^2B - rAB' \qquad (180)$$

Now we can plug in $B = K_1/A$, which implies $B' = -K_1/A^2A'$.

$$
\begin{aligned}
0 &= -2K_1 + r\frac{K_1}{A}A' + rA\frac{K_1}{A^2}A' + 2K_1A \\
0 &= 2K_1\left(-1 + r\frac{A'}{A} + A\right) \qquad (181)
\end{aligned}
$$

Which is a differential equation for $A$:

$$\frac{dA(r)}{dr} = \frac{1}{r}A(r)(1 - A(r)) \qquad (182)$$

Separating variables:

$$\frac{dA}{A(1-A)} = \frac{dr}{r} \qquad (183)$$

Integrating:

$$\ln A - \ln(1-A) = \ln r + K_2 \longrightarrow \frac{A}{1-A} = K_2 \times r \tag{184}$$

with integration constant $K_2$. And from this one has

$$A(r) = \frac{1}{1+\frac{1}{K_2 r}} \tag{185}$$

We have thus

$$ds^2 = -K_1 \left(1+\frac{1}{K_2 r}\right) c^2 dt^2 + \left(1+\frac{1}{K_2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{186}$$

For $r \longrightarrow \infty$, the metric shall be flat Minkowski space. In that limit $\left(1+\frac{1}{K_2 r}\right)^{-1} \longrightarrow 1$, and thus $K_1 = 1$. $K_2$ is determined from the weak field approximation. With $g_{00}$ from equation 106:

$$g_{00} = -1 + h_{00} = -1 - \frac{2\Phi}{c^2} = -\left(1-\frac{2GM}{rc^2}\right) = -\left(1-\frac{r_S}{r}\right) \tag{187}$$

we see that $K_2 = -c^2/(2GM)$. In the last equality we have defined the Schwarzschild radius

$$r_S = \frac{2GM}{c^2} \tag{188}$$

The Schwarzschild metric is

$$ds^2 = -\left(1-\frac{r_S}{r}\right) c^2 dt^2 + \left(1-\frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{189}$$

For an illustration: If we choose a constant time, and the equatorial plane with $\theta = \pi/2$ (which is any arbitrary plane due to the spherical symmetry), one gets for the hypersurface:

$$ds^2_{t_0,\text{plane}} = \left(1-\frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\phi^2 \tag{190}$$

This looks very similar to flat space in cylindrical coordinates

$$ds^2_{\text{flat}} = dr^2 + dz^2 + r^2 d\phi^2 = \left(1+\left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 d\phi^2 \tag{191}$$

If one chooses

$$z(r) = 2\sqrt{r_S(r-r_S)} \tag{192}$$

one can "embed" the hypersurface in flat space, illustrating the curvature of the Schwarzschild metric (figure 6, Flamm's paraboloid).

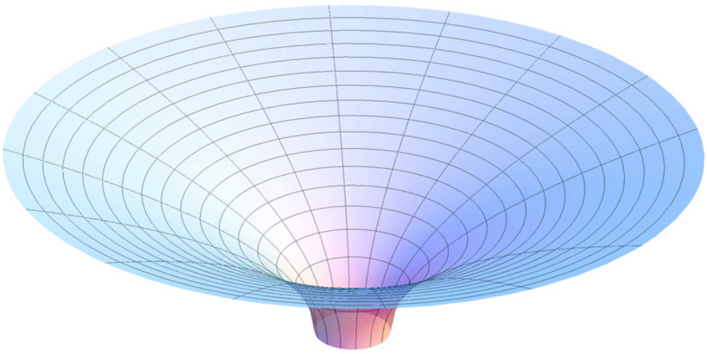

Figure 6: Flamm's paraboloid showing the curvature of the Schwarzschild metric. The 1D-radial function is rotated around the z-axis here for illustration. This is not a gravitational well. The paraboloid does not continue further down, but there is a minimum radius at $r = r_S$, where $z(r) = 2\sqrt{r_S(r - r_S)}$ gets infinitely steep, i.e. $dz/dr = \infty$. Source: Wiki