A Exercises 1

A.1 Magnitude of 4-velocity

Calculate the magnitude (vector length) of the 4-velocity u, defined by

$$u^\mu = \frac{dx^\mu}{d\tau}$$

- Use the definition of the 4-coordinates.
- Since this is a scaler, you can choose to work in any inertial frame - choose the easiest one.

Solution: Section 2.2.5

A.2 4-acceleration

Show that the 4-acceleration is perpendicular to the 4-velocity w.r.t to the metric defined

$$a^\mu = \frac{d}{d\tau}\frac{p^\mu}{m_0}$$

Solution: Equation 28

A.3 Coordinate transformation to spherical coordinates

Calculate the metric tensor of Minkowski space-time in spherical coordinates, defined by

$$\begin{aligned} x &= r\sin\theta\sin\phi \\ y &= r\sin\theta\cos\phi \\ z &= r\cos\theta \end{aligned}$$

- Calculate the differentials dx, dy, dz by deriving the transformations with respect to the new variables (r, θ, ϕ). Note that $x = x(r, \theta, \phi)$ - you will use the chain rule.
- Plug your results into the line element ds^2 of the Minkowski space-time in Cartesian coordinates.
- By inspection of the result, find the metric tensor for Minkowski space-time in spherical coordinates.

Solution: Exercise A.4 with $a = 0$, or explicitly:

$$\begin{aligned} dx &= \mathrm{s}\,\theta\,\mathrm{c}\,\phi dr + r\,\mathrm{c}\,\theta\,\mathrm{c}\,\phi d\theta - r\,\mathrm{s}\,\theta\,\mathrm{s}\,\phi d\phi \\ dy &= \mathrm{s}\,\theta\,\mathrm{s}\,\phi dr + r\,\mathrm{c}\,\theta\,\mathrm{s}\,\phi d\theta + r\,\mathrm{s}\,\theta\,\mathrm{c}\,\phi d\phi \\ dz &= \mathrm{c}\,\theta dr - r\,\mathrm{s}\,\theta d\theta \end{aligned} \tag{468}$$

$$
\begin{aligned}
& dx^2 + dy^2 + dz^2 \\
= \;& \mathrm{s}^2\theta\,\mathrm{c}^2\phi dr^2 + r^2\mathrm{c}^2\theta\,\mathrm{c}^2\phi d\theta^2 + r^2\mathrm{s}^2\theta\,\mathrm{s}^2\phi d\phi^2 \\
& +2r\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta\,\mathrm{c}^2\phi dr d\theta - 2r\,\mathrm{s}^2\theta\,\mathrm{s}\,\phi\,\mathrm{c}\,\phi dr d\phi - 2r^2\mathrm{s}\,\theta\,\mathrm{c}\,\theta\,\mathrm{s}\,\phi\,\mathrm{c}\,\phi d\theta d\phi \\
& +\mathrm{s}^2\theta\,\mathrm{s}^2\phi dr^2 + r^2\mathrm{c}^2\theta\,\mathrm{s}^2\phi d\theta^2 + r^2\mathrm{s}^2\theta\,\mathrm{c}^2\phi d\phi^2 \\
& +2r\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta\,\mathrm{s}^2\phi dr d\theta + 2r\,\mathrm{s}^2\theta\,\mathrm{s}\,\phi\,\mathrm{c}\,\phi dr d\phi + 2r^2\mathrm{s}\,\theta\,\mathrm{c}\,\theta\,\mathrm{s}\,\phi\,\mathrm{c}\,\phi d\theta d\phi \\
& +\mathrm{c}^2\theta dr^2 - 2r\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta dr d\theta + r^2\mathrm{s}^2\theta d\theta^2 \\
= \;& dr^2\left(\mathrm{s}^2\theta(\mathrm{c}^2\phi + \mathrm{s}^2\phi) + \mathrm{c}^2\theta\right) + d\phi^2\left(r^2\mathrm{s}^2\theta(\mathrm{c}^2\phi + \mathrm{s}^2\phi)\right) \\
& d\theta^2\left(r^2\mathrm{c}^2\theta(\mathrm{c}^2\phi + \mathrm{s}^2\phi) + r^2\mathrm{s}^2\theta\right) + 2r\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta(\mathrm{c}^2\phi + \mathrm{s}^2\phi - 1) dr d\theta \\
= \;& dr^2(\mathrm{s}^2\theta + \mathrm{c}^2\theta) + r^2 d\theta^2 + r^2 d\phi^2\,\mathrm{s}^2\theta \\
= \;& dr^2 + r^2 d\theta^2 + r^2\mathrm{s}^2\theta d\phi^2 = dr^2 + r^2 d\Omega \\
ds^2 = \;& -c^2dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 = -c^2dt^2 + dr^2 + r^2 d\Omega
\end{aligned}
$$

A.4 Coordinate transformation to elliptical coordinates

Calculate the metric of Minkowski space-time for the transformation to elliptical coordinates, defined by

$$
\begin{aligned}
x &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \\
y &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \\
z &= r\cos\theta
\end{aligned}
$$

For $a = 0$, this is the same as in exercise A.3. The path of calculation is the same.

Solution:

$$
\begin{aligned}
dx &= \frac{r}{\sqrt{r^2 + a^2}}\,\mathrm{s}\,\theta\,\mathrm{c}\,\phi dr + \sqrt{r^2 + a^2}\,\mathrm{c}\,\theta\,\mathrm{c}\,\phi d\theta - \sqrt{r^2 + a^2}\,\mathrm{s}\,\theta\,\mathrm{s}\,\phi d\phi \\
dy &= \frac{r}{\sqrt{r^2 + a^2}}\,\mathrm{s}\,\theta\,\mathrm{s}\,\phi dr + \sqrt{r^2 + a^2}\,\mathrm{c}\,\theta\,\mathrm{s}\,\phi d\theta + \sqrt{r^2 + a^2}\,\mathrm{s}\,\theta\,\mathrm{c}\,\phi d\phi \\
dz &= \mathrm{c}\,\theta dr - r\,\mathrm{s}\,\theta d\theta
\end{aligned}
\tag{469}
$$

$$
\begin{aligned}
& dx^2 + dy^2 + dz^2 \\
= & \frac{r^2}{r^2+a^2}\,\mathrm{s}^2\theta\,\mathrm{c}^2\phi dr^2 + (r^2+a^2)\,\mathrm{c}^2\theta\,\mathrm{c}^2\phi d\theta^2 + (r^2+a^2)\,\mathrm{s}^2\theta\,\mathrm{s}^2\phi d\phi^2 \\
& +2r\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta\,\mathrm{c}^2\phi dr d\theta - 2r\,\mathrm{s}^2\theta\,\mathrm{s}\,\phi\,\mathrm{c}\,\phi dr d\phi - 2(r^2+a^2)\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta\,\mathrm{s}\,\phi\,\mathrm{c}\,\phi d\theta d\phi \\
& +\frac{r^2}{r^2+a^2}\,\mathrm{s}^2\theta\,\mathrm{s}^2\phi dr^2 + (r^2+a^2)\,\mathrm{c}^2\theta\,\mathrm{s}^2\phi d\theta^2 + (r^2+a^2)\,\mathrm{s}^2\theta\,\mathrm{c}^2\phi d\phi^2 \\
& +2r\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta\,\mathrm{s}^2\phi dr d\theta + 2r\,\mathrm{s}^2\theta\,\mathrm{s}\,\phi\,\mathrm{c}\,\phi dr d\phi + 2(r^2+a^2)\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta\,\mathrm{s}\,\phi\,\mathrm{c}\,\phi d\theta d\phi \\
& +\mathrm{c}^2\theta dr^2 - 2r\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta dr d\theta + r^2\,\mathrm{s}^2\theta d\theta^2 \\
= & dr^2\left(\frac{r^2}{r^2+a^2}\,\mathrm{s}^2\theta(\mathrm{c}^2\phi+\mathrm{s}^2\phi)+\mathrm{c}^2\theta\right) + d\phi^2\left((r^2+a^2)\,\mathrm{s}^2\theta(\mathrm{c}^2\phi+\mathrm{s}^2\phi)\right) \\
& d\theta^2\left((r^2+a^2)\,\mathrm{c}^2\theta(\mathrm{c}^2\phi+\mathrm{s}^2\phi)+r^2\,\mathrm{s}^2\theta\right) + 2r\,\mathrm{s}\,\theta\,\mathrm{c}\,\theta(\mathrm{c}^2\phi+\mathrm{s}^2\phi-1)dr d\theta \\
= & dr^2\frac{r^2\,\mathrm{s}^2\theta + r^2\,\mathrm{c}^2\theta + a^2\,\mathrm{c}^2\theta}{r^2+a^2} + d\theta^2\left(r^2+a^2\,\mathrm{c}^2\theta\right) + d\phi^2\,\mathrm{s}^2\theta(r^2+a^2) \\
= & dr^2\frac{r^2+a^2\,\mathrm{c}^2\theta}{r^2+a^2} + d\theta^2\left(r^2+a^2\,\mathrm{c}^2\theta\right) + d\phi^2\,\mathrm{s}^2\theta(r^2+a^2) \\
ds^2 = & -c^2dt^2 + dr^2\frac{\rho^2}{r^2+a^2} + \rho^2 d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2
\end{aligned}
$$

(with $\rho^2 = r^2 + a^2\,\mathrm{c}^2\,\theta$). This will be the starting point for deriving the Kerr metric.

A.5 Lorentz transformation

A Lorentz transformation is defined by $(c\,t, x, y, z) \longrightarrow (c\,t', x', y', z')$:

$$
\begin{aligned}
t' &= \gamma\left(t - \frac{v_x}{c^2}x\right) \\
x' &= \gamma(x - v_x t) \\
y' &= y \\
z' &= z
\end{aligned}
$$

Show that a space-time interval $|s'|$, i.e. the vector length of 4-vector s', is the same as for the original vector 's' - i.e. that vector lengths are invariant under Lorentz transformations.

Solution: Equation 40

A.6 Keplerian angular velocity

Calculate for a circular orbit around mass M the angular velocity ω, which is a function of radius.

Solution: Force equilibrium

$$\begin{aligned}\frac{GM}{r^2} &= \frac{v^2}{r}\\ \frac{GM}{r^2} &= \frac{\omega^2 r^2}{r}\\ \omega &= \sqrt{\frac{GM}{r^3}}\end{aligned}$$

This will be needed for accretion disks.

A.7 Effective potential & Keplerian orbits

Re-educate yourself on Keplerian orbits in a Newtonian potential, central mass M.

- Start from $E = E_{\rm kin} + E_{\rm pot}$
- Express the velocity v in a radial and tangential component
- Introduce the angular momentum $l = r^2\dot{\phi}$
- Read off the form of the effective potential $V_{\rm eff}(r)$
- What are the turning points of the motion, as a function of E and l? ($dr/dt = 0$)
- In the energy equation, transform from $\dot{r}$ to $\frac{dr}{d\phi}$
- Transform once again to $u = 1/r$
- Take the derivative $\frac{d}{d\phi}$
- The second order differential equation for $u(\phi)$ obtained is "easy". Can you guess the solution?
- Verify that $u(\phi) = 1 + e\cos\phi$ solves the equation
- What are the turning points, as a function of l and e ?

Solution:

$$\begin{aligned}E &= \frac{1}{2}v^2 - \frac{GM}{r}\\ E &= \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r}\\ E &= \frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} - \frac{GM}{r}\\ V_{\rm eff} &= \frac{l^2}{2r^2} - \frac{GM}{r}\\ r_\pm &= \frac{-GM \pm \sqrt{G^2M^2 + 2El^2}}{2E}\end{aligned}$$

$$
\begin{aligned}
E &= \frac{1}{2}\left(\frac{dr}{d\phi}\right)^2 \dot{\phi}^2 + \frac{l^2}{2r^2} - \frac{GM}{r} \\
E &= \frac{1}{2}\left(\frac{dr}{d\phi}\right)^2 \frac{l^2}{r^4} + \frac{l^2}{2r^2} - \frac{GM}{r} \\
& u = \frac{l^2}{GM}\frac{1}{r}, \quad du = -\frac{l^2}{GM}\frac{1}{r^2}dr \\
E &= \frac{1}{2}\left(\frac{GM/l^2 r du}{d\phi}\right)^2 \frac{l^2}{r^4} + \frac{l^2 u^2 G^2 M^2}{2l^4} - \frac{G^2M^2 u}{l^2} \\
E &= \frac{G^2M^2}{l^2}\left(\frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{u^2}{2} - u\right) \\
& \text{Differentiating} \\
0 &= \frac{G^2M^2}{l^2}\left(u'u'' + u'u - u'\right) \\
0 &= u'' + u - 1 \\
u(\phi) &= 1 + e\cos(\phi - \phi_0) \\
r(\phi) &= \frac{l^2}{GM}\frac{1}{1 + e\cos(\phi - \phi_0)} \\
r_\pm &= \frac{l^2}{GM}\frac{1}{1 \pm e}
\end{aligned}
$$

This will be useful when comparing orbits around a black hole with Newtonian orbits.