B Exercises 2

B.1 Relativistic energy-momentum relation

- Take the definition of $p = (E/c, \vec{p})$ and show that the 0-component times $\vec{v}$ equals the spatial-component times c.
- Square the equality, and manipulate such that you can use γ to get rid of the v^2-term.
- Show that this yields
$$(m_0 c^2)^2 = E^2 - (|\vec{p}|c)^2$$

Solution: equation 35

B.2 Energy-Momentum tensor for perfect fluid

Show that the Energy-Momentum tensor for a perfect fluid

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu + P g^{\mu\nu}$$

get s in a local inertial frame:

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Solution:

$$\begin{aligned} u^\mu &\longrightarrow (c, 0, 0, 0) \\ g^{\mu\nu} &\longrightarrow \eta^{\mu\nu} \end{aligned}$$

Of $u^\mu u^\nu$ only the 00-component survives, so

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) \mathrm{diag}(c^2, 0, 0, 0) + P\,\mathrm{diag}(-1, 1, 1, 1) = \mathrm{diag}(\rho c^2, P, P, P) \tag{502}$$

B.3 Choice of Lagrangian

Show that the action

$$S_\tau = \int d\tau \mathcal{L}$$

with

$$\mathcal{L}(x, u) = -mc\sqrt{-g_{\mu\nu}(x) u^\mu u^\nu}$$

does not change, when one reparametrizes the proper time from τ to $\sigma(\tau)$.

Solution: section 5.3

B.4 Christoffel symbols

Let's calculate some Christoffel symbols for a static, diagonal metric.

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$$

Here, we only calculate the once which vanish with these two assumptions alone. These are:

$$\begin{array}{l}
\Gamma^{0}{}_{00},\, \Gamma^{0}{}_{11},\, \Gamma^{0}{}_{12} = \Gamma^{0}{}_{21},\, \Gamma^{0}{}_{13} = \Gamma^{0}{}_{31},\, \Gamma^{0}{}_{23} = \Gamma^{0}{}_{32},\, \Gamma^{0}{}_{22},\, \Gamma^{0}{}_{33} \\
\Gamma^{1}{}_{0i} = \Gamma^{1}{}_{i0} \text{ (with } i = 1,2,3) \\
\Gamma^{2}{}_{13} = \Gamma^{2}{}_{31},\, \Gamma^{2}{}_{0i} = \Gamma^{2}{}_{i0} \text{ (with } i = 1,2,3) \\
\Gamma^{3}{}_{12} = \Gamma^{3}{}_{21},\, \Gamma^{3}{}_{0i} = \Gamma^{3}{}_{i0} \text{ (with } i = 1,2,3)
\end{array}$$

Hence, show that these 32 of the 64 Christoffel symbols are all zero.

Solution: section 8.2

B.5 Derivative of metric tensor

Due to the metric compatibility we have $\nabla_\lambda g_{\mu\nu} = 0$. When the Christoffel symbols vanish, the covariant derivative gets the partial one, and $\partial_\lambda g_{\mu\nu} = 0$ follows directly. One can show also without using the concept of a covariant derivative, that if the Christoffel symbols vanish, the partial derivative of the metric is zero:

- Use the relation between Christoffel symbol and metric tensor
- Use the metric tensor with lower indices to lower the upper index of the Christoffel symbol, and evaluate the Kronecker-delta obtained
- Write the same equation again, with indices cyclically permuted, i.e. $i_1 \to i_3$, $i_2 \to i_1$, $i_3 \to i_2$.
- Add the two equations

Solution:

$$\begin{aligned}
\Gamma^{\lambda}{}_{\mu\nu} &= \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \\
g_{\beta\lambda}\Gamma^{\lambda}{}_{\mu\nu} &= \frac{1}{2} g_{\beta\lambda} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \\
\Gamma_{\beta\mu\nu} &= \frac{1}{2} \delta^{\alpha}_{\beta}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \\
\Gamma_{\beta\mu\nu} &= \frac{1}{2} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \\
\Gamma_{\mu\nu\beta} &= \frac{1}{2} (\partial_\nu g_{\mu\beta} + \partial_\beta g_{\mu\nu} - \partial_\mu g_{\nu\beta}) \\
0 + 0 = \Gamma_{\beta\mu\nu} + \Gamma_{\mu\nu\beta} &= \partial_\nu g_{\mu\beta}
\end{aligned}$$

B.6 Metric compatibility

Show the the metric compatibility holds for our choice of covariant derivative, i.e. that $\nabla_\lambda g_{\mu\nu} = 0$

- Start by that and use the definition of the covariant derivative
- Use the metric tensor with lower indices to lower the upper index of the Christoffel symbols
- Plug in the Christoffel symbols expressed by the metric tensor: $\Gamma_{\lambda\mu\nu} = \frac{1}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$

Solution: Equation 60

B.7 Riemann tensor

Here is an exercise in concentration... Show that the term by which two covariant derivatives do not commute is the Riemann curvature tensor.

- Start with $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A^\rho$
- First, do the inner derivatives, using the definition. This yields tensors which contain an upper and a lower index
- Then, do the outer derivatives, noting that the sign on the Christoffel symbol depends on whether one has a lower or upper index in the tensor, and that for the two indices, one gets two Christoffel terms per derivative in this step.
- From the resulting 12 terms, 8 cancel.
- The remaining four are the definition of the Riemann tensor

Solution: Beginning of section 4.1.2