D Exercises 4

D.1 The mass of Sgr A*

Derive the formula relating the mass of the central object with semi-major axis and orbital period. Then plug in some values for Sgr A*: $P = 16.0\,\mathrm{yr}$, $a = 125\,\mathrm{mas}$ (milli-arcsec), and distance $R_0 = 8.3\,\mathrm{kpc}$. You will also need the gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$. (Express the result in solar masses, $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$.)

Solution: Force equilibrium on circular orbit

$$\begin{aligned}
\frac{GM}{r^2} &= \frac{v^2}{r} \\
M &= \frac{(2\pi r)^2 r^2}{rP^2 G} = 4\pi^2 \frac{r^3}{GP^2} = 4\pi^2 \frac{(a\,R_0)^3}{GP^2} \\
&= 4\pi^2 \frac{(125/1000/3600 * \pi/180 * 8300 * 3.086 \times 10^{16}\mathrm{m})^3}{6.67 \times 10^{-11}\mathrm{m^3/s^2/kg} * (16.0 * 365.25 * 24 * 3600\mathrm{s})^2} \\
&= 8.86 \times 10^{36}\mathrm{kg} = 4.34 \times 10^6 M_\odot
\end{aligned}$$

D.2 Density

What is the density of a black hole as a function of mass, if one identifies its size with the Schwarzschild radius? At what mass is the density of a black hole smaller than that of air? (Express the result in solar masses, $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$.)

Solution: Section 10.1

D.3 Gaussian signals

Show that for a Gaussian signal

$$P_G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

The following equality holds:

$$\begin{aligned}
\langle \exp(\alpha x) \rangle &:= \int \exp(\alpha x) P_G(x) dx \\
&= \exp\left(\frac{1}{2}\alpha^2 \left\langle x^2 \right\rangle\right)
\end{aligned}$$

Solution: Equation 219

D.4 Structure function and coherence function

Show that for a real-valued function A the structure function D_A and coherence function B_A

$$\begin{aligned} D_A(\vec{r}) = D_A(\vec{r},\vec{0}) &= \left\langle |A(\vec{r}) - A(\vec{0})|^2 \right\rangle \\ B_A(\vec{r}) &= \langle A(\vec{r}) A^*(\vec{r}) \rangle \end{aligned}$$

have the simple relation

$$D_A(\vec{r}) = 2(B_A(\vec{0}) - B_A(\vec{r}))$$

Here

$$\langle Y(\vec{x}) \rangle = \int Y(\vec{x},\vec{r}) d^3 r$$

(This relation is used twice in the derivation of the Fried parameter.)

Solution: Equation 220

D.5 Radial free-fall in Newtonian gravity

Solve the equation for radial infall in Newtonian gravity to get $t(r)$.

- Start from the force equation
- Transform from $\ddot{r}$ to dv/dr.
- Solve the (very simple) differential equation to get $v(r)$.
- Fix the integration constant by demanding that at the start point r_B, $v(r_B) = 0$.
- Now use $v = dr/dt$ and separate variables to get an expression for $t(r)$.
- The integral is some work - either look it up, or try with substitutions...
- When does that particle reach $r = 0$ (as a function of r_B and M)
- Starting from $r_B = 1\,\mathrm{AU}$ for $M = 1\,M_\odot$, when does the test particle reach the center?
- The motion can also be seen as an $e = 1$ orbit with a semi-major axis of $a = r_B/2$. The time to reach the center is thus half of the the orbital period for that orbit. Does that agree with what you calculated before? It should.

Solution: Section 11.1.1