E Exercises 5

E.1 Isotropic form of metric

It is useful, to transform the Schwarzschild metric into 'isotropic' coordinates, which are closely related to Cartesian coordinates.

- Show that for standard spherical coordinates $dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega$
- We look for a new radial coordinate, ρ, that brings the metric into a form that contains $d\rho^2 + \rho^2 d\Omega$. The ansatz is
$$ds^2 = -A^2(\rho)dt^2 + B^2(\rho)\left(d\rho^2 + \rho^2 d\Omega\right)$$
Compare that to the standard coordinates. Can you read-off the coefficients of $d\Omega$? (Yes, you can...) This is a first equation for $B(\rho)$.
- Next compare the radial parts. How do they compare? You get a second equation for $B(\rho)$, including dr and $d\rho$
- Divide the two equations and simplify them, separate the two variables r and ρ.
- You now have a differential equation. One side is easy, the other needs a look-up. The integral needed is of the form
$$\int \frac{1}{\sqrt{Ax^2 + Bx + C}} dx = \frac{1}{\sqrt{A}} \ln\left[\frac{2Ax + B}{\sqrt{A}} + 2\sqrt{Ax^2 + Bx + C}\right]$$
- Don't forget to include an integration constant. For $r \to \infty$, $\rho/r \to 1$, so that in the far field there is no difference. This fixes the integration constant.
- Solve for $r(\rho)$, and get $dr/d\rho$
- Now you have all in hand to rewrite the metric.
- The result is
$$ds^2 = -\left(\frac{1 - \frac{\rho_S}{\rho}}{1 + \frac{\rho_S}{\rho}}\right)^2 c^2 dt^2 + \left(1 + \frac{\rho_S}{\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega\right)$$

Solution: Section 10.6

E.2 Vis-viva equation

For a Keplerian orbit, derive the relation between energy of the particle and semi-major axis.

- Start with the energy equation at peri-center.
- Use the definition of angular momentum, and $a = l^2/(GM) \times 1/(1 - e^2)$, and further $r_P = a(1 - e)$.

Solution: Equation 321

E.3 (No) precession in Keplerian orbits

Show that the orbit angle ϕ changes by π for a Keplerian orbit, when one goes from peri-center to apo-center.

- Start again with the energy equation
- Split up velocity in radial and tangential component, the later can be expressed by the angular momentum l.
- Go from $v_r = dr/dt$, to an equation for $d\phi/dr$. You get another term with l.
- Go to $u := 1/r$.
- $\Delta\Phi = \int_{r_P}^{r_A} \frac{d\phi}{dr}$ dr
- The integral again needs a look-up:

$$\int \frac{C^2}{\sqrt{A + 2Bc - C^2x^2}} dx = \arccos \frac{C^2/B - 1}{\sqrt{1 + AC^2/B^2}}$$

- Use $r_{A/P} = l^2/(GM) \times 1/(1 \pm e)$ and $e^2 - 1 = 2El^2/(G^2M^2)$
- You should obtain π

Solution: Included in section 11.3.3