G Exercises 7

G.1 ABCD algorithm

The beam combiner of GRAVITY measures the complex visibilities via four intensities with small phase shifts:

$$
\begin{aligned}
I_A &= I_0 + I_0 A \cos(\phi) \\
I_B &= I_0 + I_0 A \cos(\phi + \pi/2) \\
I_C &= I_0 + I_0 A \cos(\phi + \pi) \\
I_D &= I_0 + I_0 A \cos(\phi + 3\pi/2)
\end{aligned}
$$

Derive the formulas for phase ϕ and amplitude A.

Solution:

$$
\begin{aligned}
I_A &= I_0 + I_0 A \cos(\phi) \\
I_B &= I_0 + I_0 A \cos(\phi + \pi/2) = I_0 - I_0 A \sin\phi \\
I_C &= I_0 + I_0 A \cos(\phi + \pi) = I_0 - I_0 A \cos\phi \\
I_D &= I_0 + I_0 A \cos(\phi + 3\pi/2) = I_0 + I_0 A \sin\phi
\end{aligned}
$$

$$
\frac{I_D - I_B}{I_C - I_A} = \frac{2I_0 A \sin\phi}{2I_0 A \cos\phi} = \tan\phi
$$

$$
\begin{aligned}
I_A + I_B + I_C + I_D &= 4I_0 \\
(I_D - I_B)^2 + (I_C - I_A)^2 &= (2I_0 A \sin\phi)^2 + (2I_0 A \cos\phi)^2 = 4I_0^2 A^2 \\
\frac{(I_D - I_B)^2 + (I_C - I_A)^2}{(I_A + I_B + I_C + I_D)^2} &= \frac{4I_0^2 A^2}{16 I_0^2} = \frac{A^2}{4}
\end{aligned}
$$

(Equation 376)

G.2 Relativistic aberration

A light source is moving with $\vec{u} = (u_x, 0, 0)$ with respect to some rest frame. The four-vector of a photon emitted is $p = (E/c, Ev_x/c^2, Ev_y/c^2, Ev_z/c^2)$ with $v_x^2 + v_y^2 + v_z^2 = c^2$, such that $E = |\vec{p}|c$. What is the angle θ' under which the light is seen in the rest-frame, as a function of emitting angle θ and relativistic $\beta = |\vec{u}|/c$?

$$
\cos\theta' = \frac{p'_x}{|\vec{p'}|} = ...
$$

Use a Lorentz boost for expressing the primed quantities in terms of original ones, the matrix is

$$
\begin{pmatrix}
\gamma & \beta\gamma & 0 & 0 \\
\beta\gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

Solution:

$$\begin{pmatrix} E'/c \\ E'v'_x/c^2 \\ E'v'_y/c^2 \\ E'v'_z/c^2 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E/c \\ Ev_x/c^2 \\ Ev_y/c^2 \\ Ev_z/c^2 \end{pmatrix} = \begin{pmatrix} \gamma E/c + \beta\gamma Ev_x/c^2 \\ \beta\gamma E/c + \gamma Ev_x/c^2 \\ \gamma Ev_y/c^2 \\ \gamma Ev_z/c^2 \end{pmatrix} = \begin{pmatrix} \gamma E/c + \beta\gamma p_x \\ \beta\gamma E/c + \gamma p_x \\ \gamma p_y \\ \gamma p_z \end{pmatrix}$$

$$\cos\theta' = \frac{p'_x}{|\vec{p}'|} = \frac{p'_x}{E'c} = \frac{\beta\gamma E/c + p_x}{\gamma E/c + \beta\gamma p_x} = \frac{\beta|\vec{p}| + |\vec{p}|\cos\theta}{|\vec{p}| + \beta|\vec{p}|\cos\theta} = \frac{\beta + \cos\theta}{1 + \beta\cos\theta}$$

(Section 3.4)

G.3 Ricci tensor

For the Schwarzschild metric we found for the Ricci tensor

$$\begin{aligned} R_{00} &= \frac{B''}{2A} - \frac{A'B'}{4A^2} - \frac{B'^2}{4AB} + \frac{B'}{rA} \\ R_{11} &= -\frac{B''}{2B} + \frac{B'^2}{4B^2} + \frac{A'B'}{4AB} + \frac{A'}{rA} \\ R_{22} &= -\frac{1}{A} + \frac{rA'}{2A^2} + 1 - \frac{rB'}{2AB} \end{aligned}$$

What are these expressions when making the ansatz

$$\begin{aligned} B(r) &= e^{\nu(r)} \\ A(r) &= e^{\lambda(r)} \end{aligned}$$

Solution:

$$\begin{aligned} B(r) &= e^{\nu(r)} \\ B'(r) &= e^{\nu(r)}\nu'(r) = \nu'(r)B(r) \\ B''(r) &= \nu''(r)B(r) + \nu'(r)(e^{\nu(r)})\nu'(r) = B(r)(\nu''(r) + (\nu'(r))^2) \\ A(r) &= e^{\lambda(r)} \\ A'(r) &= e^{\lambda(r)}\lambda'(r) = \lambda'(r)A(r) \\ A''(r) &= \lambda''(r)A(r) + \lambda'(r)(e^{\lambda(r)})\lambda'(r) = A(r)(\lambda''(r) + (\lambda'(r))^2) \end{aligned}$$

$$\begin{aligned}
R_{00} &= \frac{B}{A}\left(\frac{B\nu''}{2B}+\frac{B\nu'^2}{2B}-\frac{A\lambda' B\nu'}{4AB}-\frac{B^2\nu'^2}{4B^2}+\frac{B\nu'}{rB}\right)\\
&= e^{\nu-\lambda}\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\lambda'\nu'}{4}+\frac{\nu'}{r}\right)\\
R_{11} &= -\frac{B\nu''}{2B}-\frac{B\nu'^2}{2B}+\frac{B\nu'^2}{4B}+\frac{A\lambda' B\nu'}{4AB}+\frac{A\lambda'}{rA}\\
&= -\frac{\nu''}{2}-\frac{\nu'^2}{4}+\frac{\lambda'\nu'}{4}+\frac{\lambda'}{r}\\
R_{22} &= 1-\frac{1}{A}\left(1-\frac{r\lambda' A}{2A}+\frac{r\nu' B}{2B}\right)\\
&= 1-e^{-\lambda}\left(1-\frac{r\lambda'}{2}+\frac{r\nu'}{2}\right)=1-e^{-\lambda}\left(1+\frac{r}{2}(\nu'-\lambda')\right)
\end{aligned}$$

(Beginning of section 13.2)

G.4 Energy-momentum tensor for a perfect fluid

For a perfect fluid:

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u_\mu u_\nu + P g_{\mu\nu}$$

The perfect fluid be at rest: $u^\alpha = (u_T, 0, 0, 0)$, and we have our ansatz for the Schwarzschild metric $g_{\mu\nu} = \mathrm{diag}(-B(r), A(r), r^2, r^2 \sin^2\theta)$, where A, B as in the previous exercise.

- From $u.u = -c^2$ derive the relation between u_T and ν.
- Evaluate $u_\alpha u_\beta$
- Write down the energy momentum tensor in terms of ν, λ, ρ, P

Solution:

$$\begin{aligned}
-c^2 &= u_\beta u^\beta = g_{\alpha\beta}u^\alpha u^\beta = g_{00}u^0u^0 = -e^\nu u_T^2\\
u_T &= e^{-\nu/2}c
\end{aligned}$$

$$u_\alpha u_\beta = u_0 u_0 = (g_{00}u^0)(g_{00}u^0) = (-e^\nu e^{nu/2}c)(-e^\nu e^{nu/2}c) = c^2 e^\nu$$

$$\begin{aligned}
T_{00} &= c^2 e^\nu \rho\\
T_{11} &= P\, g_{11} = P e^\lambda\\
T_{22} &= P\, g_{22} = P\, r^2\\
T_{33} &= P\, g_{33} = P\, r^2 \sin^2\theta
\end{aligned}$$

(Section 13.3)