H Exercises 8

H.1 Re-deriving the vacuum solution

From these expressions for the Einstein tensor, where we had used the ansatz $g = \text{diag}(-e^{\nu}, e^{\lambda}, r^2, r^2 \sin^2\theta$, can you re-derive the vacuum solution?

$$\begin{aligned} G_{00} &= \frac{e^{\nu-\lambda}}{r^2}\left(e^{\lambda} + r\lambda' - 1\right) \\ G_{11} &= = \frac{1}{r^2}\left(-e^{\lambda} + r\nu' + 1\right) \end{aligned}$$

- Start with the 00-equation. The resulting differential equation (for e^{λ}) can be solved and has an integration constant free. Name it simply C_1.
- Then do the 11-equation. You get another differential equation involving also e^{λ}, which you determined already. Here, you get another integration constant C_2.
- C_2 is fixed immediately by demanding flatness of the metric for $r \to \infty$. C_1 would require comparing the weak field approximation with Newton's gravity again, the result is simply $C_1 = r_S$.

Solution:

$$\begin{aligned} G_{00} &= 0 \\ \frac{e^{\nu-\lambda}}{r^2}\left(e^{\lambda} + r\lambda' - 1\right) &= 0 \\ 1 + r\,\lambda' e^{-\lambda} - e^{-\lambda} &= 0 \\ \left[r(1-e^{-\lambda})\right]' &= 0 \\ r(1-e^{-\lambda}) &= C_1 \\ A(r) = e^{\lambda} &= \left(1 - \frac{C_1}{r}\right)^{-1} \end{aligned}$$

$$\begin{aligned} G_{11} &= 0 \\ e^{\lambda}\left(-\frac{1}{r^2} + e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right)\right) &= 0 \\ -\frac{1}{r^2} + e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) &= 0 \\ -1 + e^{-\lambda}\,(r\,\nu' + 1) &= 0 \\ r\nu' &= e^{\lambda} - 1 \\ \nu' &= \frac{C_1}{r(r - C_1} \\ \nu &= -\ln r + \ln(r - C_1) + C_2 \\ B(r) = e^{\nu} &= \frac{1}{r}(r - C_1) \times e^{C_2} = \left(1 - \frac{C_1}{r}\right) \times e^{C_2} \end{aligned}$$

The flatness in the far field requires $C_2 = 0$.

H.2 Potential energy of a sphere

What is the gravitational potential energy of a sphere with constant density, mass M and radius R? Integrate the energy of thin shells from 0 to R.

Solution:

$$U_{\text{grav}} = -\int_0^R \frac{G\,m(r)\rho(r)}{r} 4\pi r^2 dr = -\int_0^R \frac{G\,\frac{4}{3}\pi r^3 \rho\rho}{r} 4\pi r^2 dr = -\frac{16\pi^2}{15} G\rho^2 R^5 = -\frac{3}{5}\frac{GM^2}{R}$$

See also equation 457.

H.3 Tidal forces

Consider a small body with mass m and radius r at a distance d in the gravitational field of a larger body with mass M. Calculate the tidal force as the difference between gravitational force at the closest (or furthest) point and the center of the small body. Assume $r \ll d$.

Solution:

$$\Delta F = F_{\pm} - F_0 = \frac{GMm}{(d \pm r)^2} - \frac{GMm}{d^2} = GMm\frac{d^2 - (d^2 \pm 2dr + r^2)}{(d \pm r)^2 d^2} \approx \pm \frac{2GMmr}{d^3}$$

(Equation 472)

H.4 Binary mass function

Assume a binary star with masses M_1 and M_2, orbiting the common center of mass with semi-major axes of a_1 and a_2 wtht a period P.

- From the definition of the center of mass, what is the relation between M_1, M_2, a_1, a_2 ?
- Express the total semi-major $a = a_1 + a_2$ axis in terms of M_1, M_2, a_1, a_2.
- Apply Kepler's third law for a and $M = M_1 + M_2$, and bring all masses on one side of the equation.
- Assume circular orbits, such that $v_i j = 2\pi a_j / P$ (j=1,2). Eliminate therewith the semi-major axis in the expression.
- Spectroscopically one can observe $V_j = v_j \sin i$, where i is the unknown inclination. Plug that in
- Bring all unknowns onto one side of the equation ($M_1, M_2, \sin i$). On the other side are the observables P, V. This is called the binary mass function.

Solution: Equations 474, 475, 476