# I Exercises 9

## I.1 Virial theorem

Let's derive the virial theorem for system of particles moving under each others, Newtonian gravity. Consider the scaler $S$

$$S = \sum_{k=1}^{N} \vec{p}_k.\vec{x}_k$$

For a sufficiently large, symmetric stellar system $dS/dt = 0$. Hence, calculate $dS/dt$ to arrive at $\frac{dS}{dt} = 2E_{\mathrm{kin,\,total}} + V_{\mathrm{total}}$. You will need to use these (rather obvious) relations:

- $\frac{d\vec{x}_k}{dt} = \vec{p}_k$
- $\vec{p}_k = m_k \vec{v}_k$
- $E_{\mathrm{kin},\,k} = \frac{1}{2} m_k (\vec{v}_k)^2$
- $\vec{F}_k = \frac{\vec{p}_k}{dt}$
- $\vec{F}_k = \sum_{j=1, j\neq k}^{N} \vec{F}_{jk}$
- $\vec{F}_{jk} = -\vec{F}_{kj}$
- $\vec{F}_{jk} = G \frac{m_j\, m_k}{|\vec{x}_j - \vec{x}_k|^3} (\vec{x}_j - \vec{x}_k)$

*Solution: Section 15.2*

## I.2 Inverting the Kerr metric

The Kerr metric in spherical, Boyer-Lindquist coordinates is:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{r\, r_S}{\rho^2}\right) & 0 & 0 & -\frac{a\, r_S r \sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{r^2 - r\, r_S + a^2} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{a\, r_S r \sin^2\theta}{\rho^2} & 0 & 0 & \left(r^2 + a^2 + \frac{a^2\, r_S r \sin^2\theta}{\rho^2}\right)\sin^2\theta \end{pmatrix}$$

Here, $\rho^2 = r^2 + a^2 \cos^2\theta$. Key for inverting it is the determinant of the $(t, \phi)$ sub-matrtix. Show that

$$D_{t\phi} = -\sin^2\theta\, (r^2 - r\, r_S + a^2)$$

*Solution: Equation 554*

## I.3 Event horizon(s) of the Kerr metric

As for Schwarzschild, the point at which the coefficient of the $dr^2$ term in the metric gets infinite corresponds to the event horizon. Calculate these radii. The outer is one is called $r_+$ .

*Solution:*

$$\begin{aligned} ds^2 &= -\left(1-\frac{r_S r}{\rho^2}\right)c^2dt^2 + \frac{\rho^2}{r^2 - r_S\, r + a^2}dr^2 + \rho^2 d\theta^2 \\ &\quad + \left(r^2 + a^2 + \frac{r_S r\, a^2}{\rho^2}\sin^2\theta\right)\sin^2\theta\, d\phi^2 - \frac{2 r_S r\, a \sin^2\theta}{\rho^2} c\, dt\, d\phi \end{aligned}$$

*Hence:*

$$r^2 - r_S\, r + a^2 = 0$$

*and*

$$r_\pm = \frac{r_S}{2} \pm \sqrt{\frac{r_S^2}{4} - a^2} = \frac{GM}{c^2} \pm \sqrt{\frac{G^2M^2}{c^4} - a^2}$$

## I.4 A useful relation

Show that for the Kerr metric $r_+^2 + a^2 = r_S\, r_+$ .

*Solution:*

$$r_+^2 + a^2 = \left(\frac{r_S}{2} + \sqrt{\frac{r_S^2}{4} - a^2}\right)^2 + a^2 = \frac{r_S^2}{4} + \frac{r_S}{2}\sqrt{\frac{r_S^2}{4} - a^2} + \frac{r_S^2}{4} - a^2 + a^2 = \frac{r_S^2}{2} + \frac{r_S}{2}\sqrt{\frac{r_S^2}{4} - a^2} = r_S\left(\frac{r_S}{2} + \sqrt{\frac{r_S^2}{4} - a^2}\right)$$