J Exercises 10

J.1 Kepler's law in the Kerr metric

Assume an equatorial, circular orbit in the Kerr metric: $dr = 0$, $\theta = \pi/2$ and $d\theta = 0$.

- To what simplifies the metric in this case? $ds^2 = ...$?
- Derive this equation with respect to r.
- The resulting equation can be solved for $\omega = \frac{d\phi}{dt}$, which yields thus Kepler's third law for the Kerr metric
- Verify that for $a = 0$ the result agrees with what we found for the Schwarzschild metric.

Solution:

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{r_S}{r}\right)c^2dt^2 + \left(r^2+a^2+\frac{r_S\,a^2}{r}\right)d\phi^2 - \frac{2r_S\,a}{r}c\,dt\,d\phi \\
0 &= -\frac{r_S}{r^2}c^2dt^2 + \left(2r - \frac{r_S a^2}{r^2}\right)d\phi^2 + \frac{2r_S a}{r^2}c\,dt\,d\phi \\
0 &= -c^2 r_S\left(\frac{dt}{d\phi}\right)^2 + \left(2r^3 - r_S a^2\right) + 2r_S a\,c\frac{dt}{d\phi} \\
0 &= \left(\frac{dt}{d\phi}\right)^2 - \frac{2a}{c}\frac{dt}{d\phi} - \frac{2r^3 - r_S a^2}{c^2 r_S} \\
\frac{dt}{d\phi} &= \frac{1}{2}\left(\frac{2a}{c} \pm \sqrt{\frac{4a^2}{c^2} + 4\frac{2r^3 - r_S a^2}{c^2 r_S}}\right) = \frac{a}{c} \pm \sqrt{\frac{r^3}{GM}} \\
\omega = \frac{d\phi}{dt} &= \frac{\pm\sqrt{GM}}{\sqrt{r^3} \pm a\sqrt{GM}/c}
\end{aligned}
$$

J.2 Precession of orbits in the Kerr metric

A circular, non-equatorial orbit precesses around the spin axis with an angular velocity of

$$\dot{\Omega}_{\mathrm{Kerr}} = \frac{2GJ}{c^2 r^3}$$

Evaluate this for two cases:

- A satellite orbiting Earth in 200 km height.
- The star S2 around Sgr A*, assuming a maximally spinning black hole ($\chi = 1$). The orbit of S2 has $a =$ 125 mas, and the system is located in a distance of 8.3 kpc.

Solution: Inertia of a sphere:

$$J = \frac{2}{5} M R^2 \omega = \frac{2}{5} 6 \times 10^{24} \mathrm{kg} (6378\,\mathrm{kg})^2 \frac{2\pi}{24\,h} = 7 \times 10^{34}\,\mathrm{kg\,m^2\,s^{-1}}$$

For Earth, J is lower by a small factor, as the density is higher at the center (iron core), $J = 6 \times 10^{33}\,\mathrm{kg\,m^2\,s^{-1}}$. Using that,

$$\dot{\Omega}_{\mathrm{Kerr}} = \frac{2 \times 6.67 \times 10^{-11} \mathrm{kg^{-1}\,m^3\,s^{-2}} \times 6 \times 10^{33} \mathrm{kg\,m^2\,s^{-1}}}{(3 \times 10^8 \mathrm{m\,s^{-1}})^2 \times ((6378 + 200) \times 10^3 \mathrm{m})^3} = 3.14 \times 10^{-14} \mathrm{s^{-1}}$$

(A rotation is more than 6 million years.) For Sgr A, the Schwarzschild radius is $r_S = 1.27 \times 10^{10}$ m.*

$$\dot{\Omega}_{\mathrm{Kerr}} = \frac{2 \times 6.67 \times 10^{-11} \mathrm{kg^{-1}\,m^3\,s^{-2}} \times 4.3 \times 10^6 \times 2 \times 10^{30} \mathrm{kg\,m^2\,s^{-1}} \times 1.27 \times 10^{10}\,\mathrm{m}/2}{3 \times 10^8 \mathrm{m\,s^{-1}} \times (0.125/3600 * \pi/180 \times 8300 \times 3.086 \times 10^{16} \mathrm{m})^3} = 6.5 \times 10^{-15} \mathrm{s^{-1}}$$

(A rotation is more than 30 million years.)

J.3 Derivation of ergosphere

For the derivation of the ergosphere, one needs to show that these two expressions are equivalent:

$$\begin{gathered}(r^2 - r\,r_S + a^2)(r^2 + a^2 \cos^2\theta)^2 - a^2\,r^2\,r_S^2 \sin^2\theta \\ (r^2 + a^2\cos^2\theta - rrs)((r^2 + a^2)^2 - a^2(r^2 - rrs + a^2)\sin^2\theta)\end{gathered}$$

Show that!

Solution: Expanding all and replacing the $\sin^2$ with $1 - \cos^2$:

$$\begin{aligned}&a^6 \cos^4\theta + a^4 r^2 \cos^4\theta + 2a^4 r^2 \cos^2\theta - a^4 r r_S \cos^4\theta + 2a^2 r^4 \cos^2\theta \\ &+a^2 r^4 - 2a^2 r^3 r_S \cos^2\theta + a^2 r^2 r_S^2 \cos^2\theta - a^2 r^2 r_S^2 + r^6 - r^5 r_S\end{aligned}$$

J.4 Eddington luminosity

The Eddington luminosity is the luminosity at which the radiation pressure equals the gravity on the infalling gas.

$$F_{\mathrm{grav}} = \vec{\nabla}\Phi = \frac{\sigma_{\mathrm{T}}}{c\,m_{\mathrm{P}}} F_{\mathrm{rad}}$$

Assume Newtonian gravity with the Poisson equation $\vec{\nabla}^2\Phi = 4\pi G\rho$. Evaluate

$$L_{\mathrm{edd}} = \int_S F_{\mathrm{rad}}\,dS$$

(using Gauss' law to convert the surface integral into a volume integral).

Solution: Equation 636